A Flexible Extension of the Log-Logistic Distribution with Application to Cancer Data

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Abstract: This article introduces the Type II Half Logistic Topp-Leone-G (TIIHLTL-G) family, which unifies the structural properties of the Type II Half Logistic-(G TIIHL-G) and Topp-Leone-G (TL-G) family of distributions. The novelty of the TIIHLTL-G family lies in its enhanced shape flexibility and ability to model various skewness and kurtosis patterns beyond those captured by existing extensions. The statistical features of the new TIIHLTL-G family have been thoroughly investigated, including the probability-weighted moment, hazard function, moments, order statistics, quantile function, and survival function. Parameters are estimated using classical techniques, with maximum likelihood estimation performing best overall. Application to two real cancer datasets demonstrates the superiority of the proposed model over competing distributions, including the Log-logistic and related variants, with lower AIC, and BIC confirming its improved goodness-of-fit and predictive accuracy.

Keywords: Log-logistic distribution, Topp-Leone-G family, maximum likelihood, ordinary least square, infectious disease, public health.

1. INTRODUCTION

Cancer continues to be one of the most serious global health challenges, accounting for roughly one in every five deaths worldwide [1,2]. The disease burden is rising steadily, particularly in developing nations, as populations age and exposure to risk factors such as tobacco use and pollution increases [3,4]. This growing concern has encouraged researchers to use statistical modeling as a way to understand survival behavior, predict outcomes, and support data-driven medical decision-making.

In biomedical and reliability studies, lifetime distributions are essential tools for describing the time to an event, such as equipment failure or patient survival [5,6]. Traditional models, including the Exponential, Weibull, Gamma, Rayleigh, and Log-logistic distributions are widely used but often lack the flexibility needed to describe data with strong asymmetry, heavy tails, or non-standard hazard rate shapes. These limitations have driven the development of generalized or compounded distribution families, which introduce additional shape parameters or transformation schemes to increase adaptability.

Among the many proposed families, the Type II Half Logistic-G (TIIHL-G) [7] and Topp-Leone-G (TL-G) [8] distributions have attracted notable attention. The TIIHL-G family provides greater control over skewness and kurtosis through a compounding mechanism, while the TL-G family effectively captures J-shaped hazard functions. However, each model has practical restrictions: the TIIHL-G family cannot fully accommodate non-monotonic hazard patterns, and the TL-G family cannot represent increasing, decreasing, or bathtub-shaped hazards. Other extensions, such as the Kumaraswamy Odd Logistic-G, Marshall–Olkin–Gompertz-G, and TIIHL Exponentiated-G families have improved flexibility in specific contexts but often lack a unified structure that performs consistently across diverse datasets. In recent years, there has been a significant increase in the development of generalized distribution families designed to enhance flexibility in modeling diverse types of data. A prominent framework in this regard is the T–X family [9], which has inspired

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numerous extensions in the statistical literature. For instance, the Logistic generalized distributions [10], the Kumaraswamy Odd Logistic-G [11], and the Kumaraswamy Weibull-generated family [12] have been proposed to improve tail behavior and shape control. Other related efforts include the Odd TLHL-G [13], TL-Kumaraswamy [14], Weibull Odd Fr [15], HL Odd Weibull TL-G [16], TIHL-Exponentiated-G [17], and Odd Weibull TL-G power series [18] distributions. Additionally, families such as the Marshall–Olkin-Gompertz-G [19], TL–HL Odd Lindley-G [20], TIIHL Exponentiated-G [21], Marshal–Olkin Half Logistic-G [22], Teissier-G [23], Tangent Topp-Leone [24], HL-Odd Power Generalized Weibull-G [25], Odd Kappa-G [26], Beta Topp-Leone Generated [27], Lehman Lomax [28], TIHLTL-G [29], a new odd reparameterized exponential transformed-X family of distributions [30], Marshall-Olkin cosine Topp-Leone family [31], Nitrosophic Gompertz family [32], Topp-Leone modified Kies-G family [33], and Mute generalized [34] have contributed to the ongoing expansion of flexible modeling frameworks.

More recently, further extensions have been introduced, including the Maxwell-Lomax [35], TIIETHL-Gompertz-G [36], Log-Kumaraswamy [37], Odd Fr-Weibull [38], the extension of the Kumaraswamy Exponential [39], the OBP-G [40], OBP-Logistic [41], negative binomial extension [42], modified sine [43], new modification of the Kumaraswamy generated [44], Kumaraswamy generalized inverse Lomax [45], OBP-Burr X [46],Maxwell-Burr X [47], sine exponential [48], TIHLTL exponential [49], TIHLTL-inverse Lomax [50], novel version of Maxwell [51], a new version of Gompertz [52], a new hybrid Weibull-inverse Weibull [53] distributions. These models have been widely used to illustrate improved fitting capabilities and flexibility across various data types.

However, despite these numerous contributions, many of the existing generalized families still exhibit restricted flexibility in capturing highly skewed, heavy-tailed, and complex hazard rate patterns, especially those observed in medical and survival data. Most of the aforementioned models can handle either skewness or tail heaviness, but few can simultaneously model multiple hazard rate shapes. To address these persistent limitations, this study introduces a new and more adaptable family, the TIIHLTL-G gamily of distribution, which integrates the structural strengths of the Type II Half Logistic-G and Topp-Leone-G families. The proposed model offers enhanced shape flexibility and improved data-fitting performance, making it particularly suitable for complex real-life datasets, such as cancer survival data.

A specific sub-model, the Type II Half Logistic Topp-Leone Log-logistic (TIIHLTLLLog) distribution, is also formulated as an enhanced version of the classical Log-logistic model. This extension improves the model's ability to handle heavy-tailed and right-skewed data commonly observed in medical and reliability studies. The performance of the new family is evaluated using four estimation procedures, maximum likelihood (ML), ordinary least squares (OLS), maximum product of spacing (MPS), and weighted least squares (WLS) to identify the most efficient method.

Applications to two real cancer datasets show that the proposed family provides a superior fit compared with competing distributions, as evidenced by lower Akaike (AIC), Bayesian (BIC).

The proposed TIIHLTL-G family is distinct from previously reported hybrid distributions such as the Odd TLHL-G [27], Half-Logistic Odd Weibull-Topp-Leone-G [34], and Type II Half Logistic Exponentiated-G [39] and others. While those models enhance flexibility in specific directions, they are often limited to either tail behavior or specific hazard shapes. By combining the Type II Half Logistic and Topp-Leone structures, the TIIHLTL-G family achieves a broader modeling scope—capturing multiple hazard forms (increasing, decreasing, J-shaped, and bathtub) and accommodating both skewed and heavy-tailed data. This unified formulation offers a more balanced flexibility, improving parameter interpretability and fitting accuracy across diverse datasets, especially in medical and survival contexts.

The main contributions of this work are to:

i. Develop a new compounded family (TIIHLTL-G) that bridges the flexibility gap between the TIIHL-G and TL-G models;

ii. Extend the Log-logistic model through the TIIHLTLLLog sub-model for improved modeling of skewed and heavy-tailed data; and

iii. Demonstrate, through simulation and real applications, the superior performance and adaptability of the proposed distribution.

The remainder of this paper is structured as follows: Section 2 describes the mathematical formulation and statistical properties of the TIIHLTL-G family; Section 3 discusses parameter estimation techniques; Section 4 introduces the TIIHLTLLLog sub-model; Section 5 presents simulation results; Section 6 illustrates real data applications; and Section 7 concludes the study.

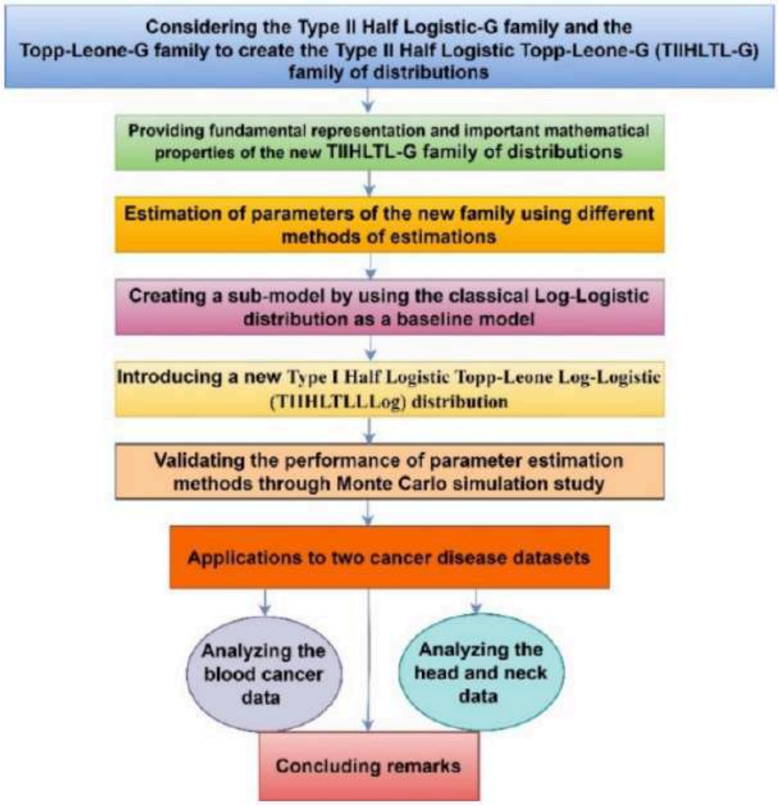

Figure 1: A pictorial flow of the study.

2.DEVELOPMENT FOR THE TYPE II HALF-LOGISTIC TOPP-LEONE-G

In this section, we employ a new family of probability distributions generated using the $T-X$ transformation technique proposed by [9], which establishes a functional link between the transformer and the transformed variable.

$$F(x)=\int_0^{W[G(x)]} r(t)\,dt\,, \tag{1}$$

Taking the derivatives of (1) provides the equivalent probability density function (pdf), which is given below:

$$f(x)=\left[\frac{d}{dx}W\left[G(x)\right]\right]rW\left[G(x)\right], \tag{2}$$

Additionally, [7] introduced the TIIHL-G characterized via its cdf and pdf:

$$F_{TIIHL-G}\left(x;\gamma,\xi\right)=\frac{2\left[1-K\left(x;\xi\right)\right]^{\gamma}}{1+\left[K\left(x;\xi\right)\right]^{\gamma}},\qquad \gamma,x>0\,, \tag{3}$$

$$f_{TIIHL-G}\left(x;\gamma,\xi\right)=\frac{2\gamma\left[K\left(x;\xi\right)\right]^{\gamma-1}k\left(x;\xi\right)}{\left[1+\left[K\left(x;\xi\right)\right]^{\gamma}\right]^{2}}\,, \tag{4}$$

As proposed in [8], the TL-G family is discussed with cdf and pdf defined as:

$$F_{TL-G}\left(x;\theta,\tau\right)=[1-(1-H(x;\tau))^2]^{\theta}\,, \tag{5}$$

and

$$f_{TL-G}\left(x;\theta,\tau\right)=2\theta h(x;\tau)[1-h(x;\tau)][(1-H(x;\tau))^2]^{\theta-1}\,, \tag{6}$$

here $\theta>0$ represent the shape parameter,

Assume the Topp-Leone-G is the baseline family, then the TIIHLTL-G family of distribution can be derived by replacing $r(t)$ and $W[G(x)]$ in (1). Thus

$$F_{TIIHLTL-G}\left(x;\gamma,\theta,\tau\right)=2\gamma k\int_0^{[1-(1-H(x;\tau))^2]^{\theta}}\frac{\left(x;\xi\right)\left[K\left(x;\xi\right)\right]^{\gamma-1}}{\left[1+\left[K\left(x;\xi\right)\right]^{\gamma}\right]^{2}}dx$$

$$=\frac{2[1-(1-H(x;\tau))^2]^{\theta\gamma}}{1+[1-(1-H(x;\tau))^2]^{\theta\gamma}};\qquad -\infty<x<\infty\,, \tag{7}$$

where $\gamma>0,\theta>0$ are the shape parameters, $H(x;\tau)$ is the cdf of the parent model and τ being the vector parameter of the parent model.

The TIIHLTL-G's pdf is subsequently determined from equation (7) as

$$f_{TIIHLTL-G}\left(x;\gamma,\theta,\tau\right)=\frac{4\theta\gamma(1-H(x;\tau)h(x;\tau)\,[1-(1-H(x;\tau))^2]^{(\theta-1)}[1-(1-H(x;\tau))^2]^{\theta(\gamma-1)}}{\left[1+[1-(1-H(x;\tau))^2]^{\theta\gamma}\right]^2}. \tag{8}$$

where $h(x;\tau)$ being the pdf of the parent model.

2.1. Mixture Representation

Considering the expansion of the generalized binomial as:

$$(1+k)^{-b}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b+i)}{i!\Gamma(i)}k^i\,. \quad (9)$$

Employing (9), the pdf in (8) can be expressed as

$$f_{TIIHLTL-G}(x;\gamma,\theta,\tau)=4\theta\gamma(1-H(x;\tau)h(x;\tau)$$
$$\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(2+i)}{i!\Gamma(i)}\left(1-(1-H(x;\tau))^2\right)^{i\theta\gamma+\theta(\gamma-1)+\theta-1}. \quad (10)$$

Considering binomial expression for $|k|<1$ and $b>0$, then we can obtain

$$(1-k)^b=\sum_{j=0}^{\infty}\binom{b}{j}(-1)^jk^j\,. \quad (11)$$

Using (11), the pdf defined in (10) will become

$$f_{TIIHLTL-G}(x;\gamma,\theta,\tau)=4\theta\gamma h(x;\tau)\sum_{i=0}^{\infty}\frac{(-1)^{i+j}\Gamma(2+i)}{i!\Gamma(i)}$$
$$\sum_{j=0}^{\infty}\binom{\theta\gamma(i+1)-1}{j}\left(1-H(x;\tau)\right)^{2j+1}$$

$$=4\theta\gamma h(x;\tau)\sum_{i,j,l=0}^{\infty}\mathrm{A}_{i,j,l}\left(H(x;\tau)\right)^l. \quad (12)$$

which is the pdf of the TIIHLTL-G family, where

$$\mathrm{A}_{i,j,l}=\frac{(-1)^{i+j+l}\Gamma(2+i)}{i!\Gamma(i)}\binom{\theta\gamma(i+1)-1}{j}\binom{2j+1}{l}.$$

2.2. Statistical Properties of the TIHLTL-G Family

This section examines various statistical properties of the TIIHLTL-G family.

2.2.1 Probability Weighted Moment (PWM)

The PWMs denoted by $\delta_{r,s}$, is presented as:

$$\delta_{r,s}=\int_{-\infty}^{\infty}x^r\left(F(x)\right)^sf(x)dx;\quad -\infty<x<\infty, \quad (13)$$

where $F(x)$ and $f(x)$ are provided in equations (7) and (8).

Let us simplify the term $\left(F(x)\right)^sf(x)$ as:

$$f(x)\left(F(x)\right)^s=\frac{4\theta\gamma(1-H(x;\tau)h(x;\tau)[1-(1-H(x;\tau))^2]^{(\theta-1)}[1-(1-H(x;\tau))^2]^{\theta(\gamma-1)}}{\left[1+[1-(1-H(x;\tau))^2]^{\theta\gamma}\right]^2}$$
$$\left\{\frac{2[1-(1-H(x;\tau))^2]^{\theta\gamma}}{1+[1-(1-H(x;\tau))^2]^{\theta\gamma}}\right\}^s$$

$$=\frac{2^{2+s}\theta\gamma(1-H(x;\tau)h(x;\tau)[1-(1-H(x;\tau))^2]^{\theta(\gamma-1)+(\theta-1)+s}}{\left[1+[1-(1-H(x;\tau))^2]^{\theta\gamma}\right]^{2+s}} \quad (14)$$

Based on (9), (14) can be provided as:

$$f(x)\left(F(x)\right)^s=2^{2+s}\theta\gamma(1-H(x;\tau)h(x;\tau)$$
$$\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(2+s+i)}{i!\Gamma(i)}\left(1-(1-H(x;\tau))^2\right)^{i\theta\gamma+\theta(\gamma-1)+(\theta-1)+s}$$

$$=2^{2+s}\theta\gamma h(x;\tau)\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(2+s+i)}{i!\Gamma(i)}$$
$$\sum_{j=0}^{\infty}\binom{\theta\gamma(i+1)+s-1}{j}(-1)^j\left(1-H(x;\tau)\right)^{2j+1}$$

$$=h(x;\tau)\sum_{i,j,l=0}^{\infty}B_{i,j,l}H(x;\tau)^l\,, \quad (15)$$

where

$$B_{i,j,l}=\frac{2^{2+s}\theta\gamma(-1)^{i+j+l}\Gamma(2+s+i)}{i!\Gamma(i)}\binom{\theta\gamma(i+1)+s-1}{j}\binom{2j+1}{l}$$

Inserting (15) into (13) gives the PWM for the proposed family of distributions.

$$\delta_{r,s}=\int_{-\infty}^{\infty}x^rh(x;\tau)\sum_{i,j,l=0}^{\infty}B_{i,j,l}H(x;\tau)^l\,dx=\sum_{i,j,l=0}^{\infty}\Delta_{i,j,l}, \quad (16)$$

where $\Delta_{i,j,l}=B_{i,j,l}\int_{-\infty}^{\infty}x^rh(x;\tau)H(x;\tau)^l\,dx$.

2.2.2. Moments

The r^{th} moment for the novel family by utilizing (12) as follows:

$$\upsilon_r'=\int_{-\infty}^{\infty}f(x)x^rdx\,,\ =\sum_{i,j,l=0}^{\infty}\Phi_{i,j,l}\,, \quad (17)$$

where $\Phi_{i,j,l}=\mathrm{A}_{i,j,l}4\theta\gamma\int_{-\infty}^{\infty}x^rh(x;\tau)H(x;\tau)^ldx$.

2.2.3. Moment Generating Function (MGF)

The MGF of the TIIHLTL-G family might be obtained by considering (12) as follows:

$$M_x(t)=\int_{-\infty}^{\infty} f(x)e^{tx}dx\,,$$

$$=4\theta\gamma\int_{-\infty}^{\infty} h(x;\tau)\sum_{i,j,l=0}^{\infty} A_{i,j,l}\left(H(x;\tau)\right)^{l} e^{tx}\,dx$$

$$=\sum_{i,j,l=0}^{\infty}\Psi_{i,j,l}\,, \tag{18}$$

where $\Psi_{i,j,l}=4\theta\gamma h(x;\tau)A_{i,j,l}\int_{-\infty}^{\infty}\left(H(x;\tau)\right)^{l} e^{tx}\,dx$.

2.2.4. Reliability Function

The reliability function, also known as the survival function, is frequently used to estimate the likelihood that a patient will live longer than a given period.

$$R(x;\gamma,\theta,\tau)=1-\left\{2\left[1-M(x;\tau)\right]^{\theta\gamma}\left[1+\left[1-M(x;\tau)\right]^{\theta\gamma}\right]^{-1}\right\}, \tag{19}$$

where $M(x;\tau)=\left(1-H(x;\tau)\right)^{2}$.

2.2.5. Hazard Function (HF)

The HF considered as the conditional failure rate is explained according to (8) and (19):

$$\eta(x;\gamma,\theta,\tau)=\frac{f(x;\gamma,\theta,\tau)}{R(x;\gamma,\theta,\tau)}\,,$$

$$\eta(x;\gamma,\theta,\tau)=\frac{4\theta\gamma(1-H(x;\tau)h(x;\tau)[1-M(x;\tau)]^{(\theta-1)+\theta(\gamma-1)}}{\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]^{2}\left\{1-\left(\frac{2[1-M(x;\tau)]^{\theta\gamma}}{1+[1-M(x;\tau)]^{\theta\gamma}}\right)\right\}}$$

$$=\frac{4\theta\gamma h(x;\tau)\left(M(x;\tau)\right)^{\frac{1}{2}}[1-M(x;\tau)]^{\theta\gamma-1}}{\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]\left\{1-[1-M(x;\tau)]^{\theta\gamma}\right\}}\,. \tag{20}$$

2.2.6. Order Statistics (OS)

OS plays a fundamental role in various statistical domains, notably in fields like survival analysis as well as life testing, where their application has been substantial. Consider a random variable $X_1,X_2,\ldots,X_n$ which is independent and identically distributed, then the pdf of the r^{th} OS is expressed as:

$$f_{r,n}(x)=\left[B(r,n-r+1)\right]^{-1}[1-F(x)]^{n-r}F_x^{r-1}f(x)\,. \tag{21}$$

Replacing the cdf and pdf (21) with (7) and (8), we have the following:

$$f_{r,n}(x;\gamma,\theta,\tau)=\left\{\frac{2[1-M(x;\tau)]^{\theta\gamma}}{1+[1-M(x;\tau)]^{\theta\gamma}}\right\}^{r-1}\left\{\frac{1-[1-M(x;\tau)]^{\theta\gamma}}{1+[1-M(x;\tau)]^{\theta\gamma}}\right\}^{n-1}$$

$$\times\;\frac{4\theta\gamma\left[B(r,n-r+1)\right]^{-1}(1-H(x;\tau)h(x;\tau)[1-M(x;\tau)]^{\theta\gamma-1}}{\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]^{2}}$$

$$=\frac{\theta\gamma 2^{1+r}\left\{1-[1-M(x;\tau)]^{\theta\gamma}\right\}^{n-1}h(x;\tau)\left(M(x;\tau)\right)^{\frac{1}{2}}\left(1-M(x;\tau)\right)^{\theta\gamma r-1}}{B(r,n-r+1)\left\{1+[1-M(x;\tau)]^{\theta\gamma}\right\}^{r+n}}\,, \tag{22}$$

hence, (22) is regarded as the r^{th} OS of the TIIHLTL-G family.

2.2.7. Quantile Function (QF)

To attain the QF of the TIIHLTL-G family, we invert the TIIHLTL-G's cdf defined in (7) as

$$u=\frac{2[1-(1-H(x;\tau))^{2}]^{\theta\gamma}}{\left\{1+[1-(1-H(x;\tau))^{2}]^{\theta\gamma}\right\}}\,. \tag{23}$$

As a result, the QF for the TIIHLTL-G is determined by simplifying (23) as:

$$Q(x)=\left[1-\left[1-\left(\frac{u}{2-u}\right)^{\frac{1}{\theta\gamma}}\right]^{\frac{1}{2}}\right]H^{-1}(x;\tau)\,. \tag{24}$$

3. METHODS OF PARAMETER ESTIMATION

This section introduces estimate methods from the TIIHLTL-G family and investigates four methods of estimation.

3.1. Maximum Likelihood Estimation (MLE)

The MLE is used in the study due to its non-complex approach. To find the parameters using the MLE technique, the log-likelihood function for the TIIHLTL-G family is as follows:

$$\begin{aligned}\log(L)&=n\log 4+n\log\gamma+n\log\theta+\\&\sum_{i=0}^{n}\log(1-H(x;\tau))+(\theta-1)\sum_{i=0}^{n}\log\left[1-[1-M(x;\tau)]\right]+\\&\theta(\gamma-1)\sum_{i=0}^{n}\log\left[1-[1-M(x;\tau)]\right]-2\\&\sum_{i=0}^{n}\log\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]+\sum_{i=0}^{n}\log h(x;\tau)\end{aligned}\,. \tag{25}$$

Differentiating (25) for the parameters involved, and setting it to zero, we can obtain the estimates of the distribution. The derivatives of the parameters $\hat{\gamma}_{mle},\hat{\theta}_{mle}$ and $\hat{\tau}_{mle}$ are presented as follows:

$$\frac{\delta \log L}{\delta \gamma} = \frac{n}{\gamma} + \sum_{i=0}^{n} \theta \log\left[1-[1-M(x;\tau)]\right] - 2\sum_{i=0}^{n} \log\left[1+[1-M(x;\tau)]^{\theta\gamma}\right] \times \sum_{i=0}^{n} \frac{\log\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]}{\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]} = 0 . \quad (26)$$

$$\frac{\delta \log L}{\delta \gamma} = \frac{n}{\theta} + \sum_{i=0}^{n} \gamma \log\left[1-[1-M(x;\tau)]\right] - 2\sum_{i=0}^{n} \log\left[1+[1-M(x;\tau)]^{\theta\gamma}\right] \times \sum_{i=0}^{n} \frac{\log\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]}{\left[1+[1-M(x;\tau)]^{\theta\gamma}\right]} = 0 . \quad (27)$$

$$\frac{\delta \log L}{\delta \tau} = \sum_{i=0}^{n} \frac{h(x;\tau)^{\tau}}{h(x;\tau)} + 2\sum_{i=0}^{n} \frac{(\gamma-1)\theta(1-H(x;\tau)H(x;\tau)^{\tau}}{\left[1-[1-M(x;\tau)]\right]} - 2\sum_{i=1}^{n} \frac{2\gamma\theta[1-M(x;\tau)]^{\gamma\theta-1}M(x;\tau)H(x;\tau)^{\tau}}{\left[1+[1-M(x;\tau)]^{\gamma\theta}\right]} = 0 . \quad (28)$$

3.2. Maximum Product of Spacing (MPS)

Considering $x_1, x_2, x_3, ..., x_m$ is randomly chosen from the TIIHLTL-G family with cdf $R(x;\gamma,\theta,\tau)$, and order sample as $x_{(1)},...,x_{(m)}$. The spacing

$A_p = R(x_{(p)}) - R(x_{(p-1)})$ for $p = 1,2,3...,m$, where and $R(x_{(m+1)}) = 1$,

Therefore,

$$R(x_{(p)};\gamma,\theta,\tau) = \frac{2[1-(1-K(x_{(p)};\gamma,\theta,\tau))^2]^{\theta\gamma}}{1+[1-(1-K(x_{(p)};\gamma,\theta,\tau))^2]^{\theta\gamma}},$$

and

$$R(x_{(p-1)};\gamma,\theta,\tau) = \frac{2[1-(1-K(x_{(p-1)};\gamma,\theta,\tau))^2]^{\theta\gamma}}{1+[1-(1-K(x_{(p-1)};\gamma,\theta,\tau))^2]^{\theta\gamma}}.$$

Then,

$$A_p = R(x_{(p)}) - R(x_{(p-1)}) = \frac{2[1-(1-K(x_{(p)};\gamma,\theta,\tau))^2]^{\theta\gamma}}{1+[1-(1-K(x_{(p)};\gamma,\theta,\tau))^2]^{\theta\gamma}} - \frac{2[1-(1-K(x_{(p-1)};\gamma,\theta,\tau))^2]^{\theta\gamma}}{1+[1-(1-K(x_{(p-1)};\gamma,\theta,\tau))^2]^{\theta\gamma}}, \quad (29)$$

and

$$Z(x;\gamma,\theta,\tau) = \frac{1}{m+1}\sum_{p=1}^{m+1} \log A_p . \quad (30)$$

Differentiating $Z(x;\gamma,\theta,\tau)$ for the parameters that are involved, setting it to zero, and solving the nonlinear equations produces parameter estimates for $\hat{\gamma}_{mps}, \hat{\theta}_{mps}$ and $\hat{\tau}_{mps}$:

$$\frac{\partial Z(x;\gamma,\theta,\tau)}{\partial \gamma} = \frac{1}{m+1}\sum_{p=1}^{m+1} \frac{1}{A_p}\left[A_1(x_{(p)};\gamma,\theta,\tau) - A_2(x_{(p-1)};\gamma,\theta,\tau)\right] = 0 , \quad (31)$$

$$\frac{\partial Z(x;\gamma,\theta,\tau)}{\partial \theta} = \frac{1}{m+1} \sum_{p=1}^{m+1} \frac{1}{A_p}\left[B_1(x_{(p)};\gamma,\theta,\tau) - B_2(x_{(p-1)};\gamma,\theta,\tau)\right] = 0 , \quad (32)$$

and

$$\frac{\partial Z(x;\gamma,\theta,\tau)}{\partial \tau} = \frac{1}{m+1} \sum_{p=1}^{m+1} \frac{1}{A_p}\left[C_1(x_{(p)};\gamma,\theta,\tau) - C_2(x_{(p-1)};\gamma,\theta,\tau)\right] = 0 . \quad (33)$$

3.3. Ordinary Least Square Estimator(OLSE)

The OLS parameter estimation approach was proposed by [54]. Consider $X_{(1)} \le ... \le X_{(m)}$ to be the order statistics of a random sample obtained from any probability distribution. Then the mean and variance of the i^{th}-order statistic is expressed as follows:

$E\left[T(X_{(i)})\right] = \frac{i}{m+1}$ and the

$\text{var}\left[T(X_{(i)})\right] = \frac{i(m+1-i)}{(m+1)^2(m+2)}$ $\forall_i \in 1,2,3,...,n$.

The function $D = \sum_{i=1}^{m}\left[T(X_{(i)}) - \Upsilon(i)\right]^2$ can be minimized for parameters in the distribution to obtain the OLS estimator, where $T(X_{(i)})$ and $\Upsilon(i)$ represent the theoretical cdf of the observation $(X_{(i)})$ obtained from the underlying distribution, and empirical cdf obtained by $\Upsilon(i) = \frac{i}{m+1}$ respectively.

The function

$$D = \sum_{i=1}^{m}\left[T(X_{(i)}) - \Upsilon(i)\right]^2 , \quad (34)$$

can be obtained for the TIIHLTL-G familyby substituting the theoretical and the empirical cdf appropriately as given below and differentiating for the three parameters γ,θ,τ then set it to zero, solve the equations, and will givesparameter estimates of $\hat{\gamma}_{ols}, \hat{\theta}_{ols}$ and $\hat{\tau}_{ols}$.

$$D(x;\gamma,\theta,\tau)=\sum_{i=1}^{m}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}, \quad (35)$$

$$\frac{\partial D(x;\gamma,\theta,\tau)}{\partial(\gamma)}=\frac{\partial}{\partial(\gamma)}\sum_{i=1}^{m}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}=0, \quad (36)$$

$$\frac{\partial D(x;\gamma,\theta,\tau)}{\partial(\theta)}=\frac{\partial}{\partial(\theta)}\sum_{i=1}^{m}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}=0, \quad (37)$$

$$\frac{\partial D(x;\gamma,\theta,\tau)}{\partial(\tau)}=\frac{\partial}{\partial(\tau)}\sum_{i=1}^{m}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}=0, \quad (38)$$

3.4. Weighted Least Square Estimator (WLSE)

The WLS parameter estimation approach was also proposed by [54]. Consider $X_{(1)}...X_{(m)}$ to represent the order statistics of a random sample chosen according to a probability distribution. The mean and variance of the i^{th} order statistic is provided as follows:

$E\left[T\left(X_{(i)}\right)\right]=\frac{i}{m+1}$ and the $\operatorname{var}\left[T\left(X_{(i)}\right)\right]=\frac{i(m+1-i)}{(m+1)^{2}(m+2)}$ $\forall_{i}\in 1,2,3,...,m$,

the function

$$W=\sum_{i=1}^{m}w_{i}\left[T\left(X_{(i)}\right)-\Upsilon(i)\right]^{2}, \quad (39)$$

can be minimized for parameters in the distribution to obtain the WLS estimator, where $T\left(X_{(i)}\right)$ and $\Upsilon(i)$ represent the theoretical cdf of the observation $\left(X_{(i)}\right)$ obtained from the underlying distribution, and empirical cdf obtained by $\Upsilon(i)=\frac{i}{m+1}$ respectively and $w_{i}=\frac{(m+1)(m+2)}{i(m+1-i)}$.

The function $W=\sum_{i=1}^{m}w_{i}\left[T\left(X_{(i)}\right)-\Upsilon(i)\right]^{2}$ can be obtained for the TIIHLTL-G family by substituting the theoretical and the empirical cdf appropriately as given below and differentiate with respect to the three parameters γ,θ,τ, then set it to zero solve the equations, will produce parameter estimates of $\hat{\gamma}_{wls},\hat{\theta}_{wls}$ and $\hat{\tau}_{wls}$:

$$W(x;\gamma,\theta,\tau)=\sum_{i=1}^{m}w_{i}\frac{(m+1)(m+2)}{i(m+1-i)}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}, \quad (40)$$

$$\frac{\partial W(x;\gamma,\theta,\tau)}{\partial(\gamma)}=\frac{\partial}{\partial(\gamma)}\sum_{i=1}^{m}w_{i}\frac{(m+1)(m+2)}{i(m+1-i)}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}=0, \quad (41)$$

$$\frac{\partial W(x;\gamma,\theta,\tau)}{\partial(\theta)}=\frac{\partial}{\partial(\theta)}\sum_{i=1}^{m}w_{i}\frac{(m+1)(m+2)}{i(m+1-i)}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{m+1}\right)^{2}=0, \quad (42)$$

$$\frac{\partial W(x;\gamma,\theta,\tau)}{\partial(\tau)}=\frac{\partial}{\partial(\tau)}\sum_{i=1}^{n}w_{i}=\frac{(n+1)(n+2)}{i(n+1-i)}\left(\frac{2\left[1-M(x;\tau)\right]^{\theta\gamma}}{1+\left[1-M(x;\tau)\right]^{\theta\gamma}}-\frac{i}{n+1}\right)^{2}=0. \quad (43)$$

4. SUB-MODEL

We used the Log-logistic distribution as a parent distribution to develop a novel version termed the TIIHLTL-Log-logistic (TIIHLTL-LLog) distribution.

4.1. The TIIHLTL-LLog Distribution

The TIIHLTL-LLog aims to improve the Log-Logistic distribution, hence the Log-Logistic model with parameter τ are stated as:

$$J(x;\tau)=\left(1+x^{\tau}\right)^{-1}x^{\tau};\ x>0,\ \tau>0, \quad (44)$$

and

$$j(x;\tau)=\left(1+x^{\tau}\right)^{-2}\tau x^{\tau-1}, \quad (45)$$

where τ a scale parameter.

The proposed cdf and pdf of the TIIHLTL-LLog distribution with parameters θ,γ and τ are obtained and expressed as follows:

By substituting the cdf of the Log-logistic into (7), we have:

$$F_{TIIHLTLLLog}(x;\gamma,\theta,\tau)=2\left(1-(1-\left(1+x^{\tau}\right)^{-1}x^{\tau})^{2}\right)^{\theta\gamma}\left(1+[1-(1-\left(1+x^{\tau}\right)^{-1}x^{\tau})^{2}]^{\theta\gamma}\right)^{-1},\ x>0;\gamma,\theta,\tau>0. \quad (46)$$

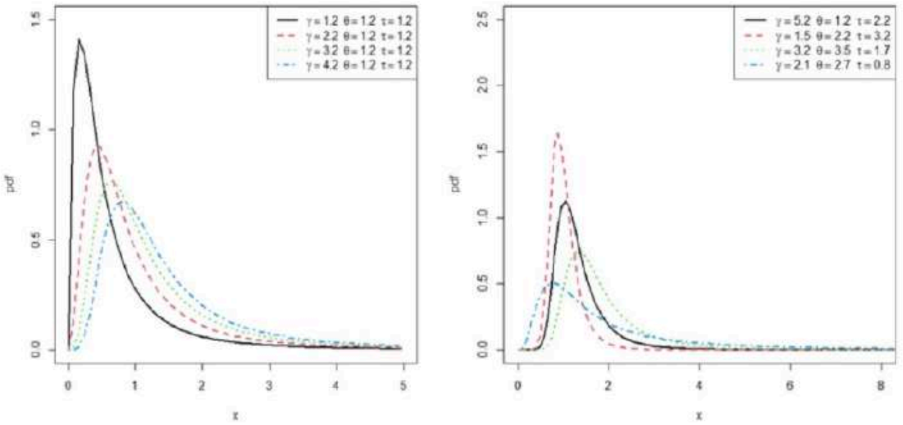

Figure 2: Probability density functions of the TIIHLTLLLog distribution for selected parameter values. The plots illustrate the model's flexibility in capturing diverse shapes from symmetric to highly right-skewed forms—depending on parameter combinations.

The pdf of the TIIHLTL-LLog is obtained by utilizing (8) as:

$$f\left(x;\gamma,\theta,\tau\right)=4\gamma\theta\tau x^{\tau-1}\left(1+x^{\tau}\right)^{-1}\left(1-(1+x^{\tau})^{-2}\right)^{(\theta-1)}$$
$$\left(1-(1+x^{\tau})^{-2}\right)^{\theta(\gamma-1)}\left(1+\left(1-(1+x^{\tau})^{-2}\right)^{\theta\gamma}\right)^{-2}\left(1+x^{\tau}\right)^{-2}, \tag{47}$$

where γ,θ are the shape parameters and the τ being the scale parameter.

Figures **2** and **3** depict the pdf and hazard functions for the novel TIIHLTL-LLog distribution. Figure **2** illustrates the pdf with right-skewed shapes across different parameter combinations. while, Figure **3** showcases the hazard function, displaying diverse shapes which include increasing, upside-down bathtub, and decreasing shapes.

Figure **2** shows that varying the shape parameters significantly alter the distribution's skewness and tail behavior. For smaller parameter values, the curve is right-skewed, while larger values yield more symmetric shapes. This demonstrates the model's ability to accommodate datasets with varying degrees of asymmetry and tail heaviness.

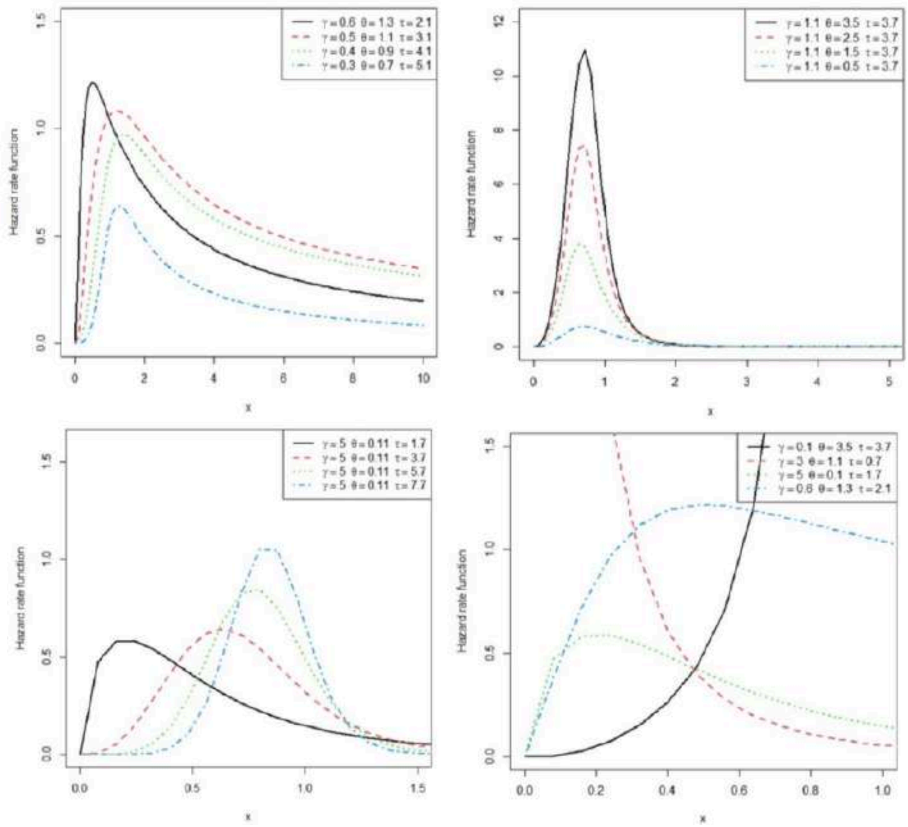

Figure 3: Hazard rate functions of the TIIHLTLLLog distribution under different parameter settings, displaying increasing, decreasing, and bathtub-shaped hazard patterns.

As shown in Figure **3**, the TIIHLTLLLog distribution captures several hazard shapes, including monotonic and non-monotonic forms. This flexibility is essential for modeling survival and reliability data where hazard rates can vary across time

5. SIMULATION STUDIES

The performance of four parameter estimation methods, namely ML, MPS, OLS, and WOLS, is systematically examined concerning the parameters of the TIIHLTL-LLog distribution. The parameters were set to be γ =1.2,θ =2.3, and τ =1.5 on account of 10,000 replications. The sample sizes (n) utilized are 10,30,70,100,200,500,1000 and 2500.

Table **1** reports the estimates, biases, and MSE obtained through simulation studies for the parameter estimation using the MLE and MPS methods.

Table **2** provides a detailed overview of the simulation study results on estimates, biases, and MSE derived from the parameter estimation process using the OLSE and WLSE methods.

Table **3** provides a concise summary of the simulation study, focusing on the four estimation methods, their corresponding sample sizes, Mean Square Error (MSE), and the cumulative rank scores ($\sum Ranks$). This table appears as a comprehensive reference for evaluating the performance of the estimation methods across distinct sample sizes.

Table **4** presents a summary of the overall ranking based on the Mean Squared Error (MSE) for the four methods of estimation. The MLE achieved the best performance with an overall MSE rank score of 12.5, securing the top position with a final rank of 1. Following closely is the MPS with an overall rank score of 16, earning the second position. The WLSE secured the third position with an overall rank score of 20. Meanwhile, the LSE had an overall rank score of 30, placing it in the fourth and final position in terms of MSE performance. These results offer valuable insights into the relative efficacy of each estimation approach in capturing the underlying parameters of the TIIHLTL-LLog distribution.

Table 1: Simulation Study on Parameter Estimation using MLE and MPE Methods

Sample size (n)	Measures	MLE			MPS		
		γ =1.2	θ =2.3	τ =1.5	γ =1.2	θ =2.3	τ =1.5
10	Estimates	1.1839	2.2570	1.9462	1.1974	2.4515	1.4299
	Biases	-0.0161	-0.0430	0.4462	-0.0026	0.1515	-0.0701
	MSE	0.1333	0.1777	1.4068	0.1340	0.1164	0.4569
30	Estimates	1.1585	2.3490	1.6076	1.1619	2.4308	1.4367
	Biases	-0.0415	0.0490	0.1076	-0.0381	0.1308	-0.0633
	MSE	0.0463	0.0375	0.1163	0.0481	0.0414	0.0888
70	Estimates	1.1555	2.3710	1.5473	1.1610	2.4089	1.4598
	Biases	-0.0445	0.0710	0.0473	-0.0390	0.1089	-0.0402
	MSE	0.0225	0.0183	0.0421	0.0240	0.0216	0.0352
100	Estimates	1.1588	2.3762	1.5311	1.1595	2.4063	1.4666
	Biases	-0.0412	0.0762	0.0311	-0.0405	0.1063	-0.0334
	MSE	0.0162	0.0135	0.0245	0.0178	0.0182	0.0239
200	Estimates	1.1575	2.3803	1.5147	1.1593	2.3952	1.4779
	Biases	-0.0425	0.0803	0.0147	-0.0407	0.0952	-0.0221
	MSE	0.0085	0.0113	0.0120	0.0092	0.0138	0.0118
500	Estimates	1.1604	2.3791	1.5034	1.1621	2.3834	1.4887
	Biases	-0.0396	0.0791	0.0034	-0.0379	0.0834	-0.0113
	MSE	0.0040	0.0089	0.0044	0.0043	0.0097	0.0045
1000	Estimates	1.1606	2.3745	1.5024	1.1649	2.3755	1.4934
	Biases	-0.0394	0.0745	0.0024	-0.0351	0.0755	-0.0066
	MSE	0.0027	0.0077	0.0020	0.0026	0.0079	0.0022
2500	Estimates	1.1618	2.3738	1.5020	1.1658	2.3716	1.4975
	Biases	-0.0382	0.0738	0.0020	-0.0342	0.0716	-0.0025
	MSE	0.0020	0.0071	0.0009	0.0017	0.0067	0.0009

Table 2: Simulation Study on Parameter Estimation using OLSE and WLSE Methods

Sample size (n)	Measures	OLSE			WLSE		
		γ =1.2	θ =2.3	τ =1.5	γ =1.2	θ =2.3	τ =1.5
10	Estimates	1.1944	.3522	1.7633	1.1809	2.4293	1.5651
	Biases	-0.0056	0.0522	0.2633	-0.0191	0.1293	0.0651
	MSE	0.1691	0.1542	0.8176	0.1455	0.1203	0.6112
30	Estimates	1.1646	2.3806	1.5375	1.1635	2.3819	1.5308
	Biases	-0.0354	0.0806	0.0375	-0.0365	0.0819	0.0308
	MSE	0.0541	0.0519	0.1432	0.0512	0.0492	0.1203
70	Estimates	1.1592	2.3837	1.5107	1.1590	2.3783	1.5192
	Biases	-0.0408	0.0837	0.0107	-0.0410	0.0783	0.0192
	MSE	0.0243	0.0229	0.0487	0.0232	0.0191	0.0427
100	Estimates	1.1605	2.3841	1.5048	1.1602	2.3787	1.5126
	Biases	-0.0395	0.0841	0.0048	-0.0398	0.0787	0.0126
	MSE	0.0171	0.0199	0.0319	0.0161	0.0161	0.0277
200	Estimates	1.1580	2.3867	1.5007	1.1546	2.3896	1.5067
	Biases	-0.0420	0.0867	0.0007	-0.0454	0.0896	0.0067
	MSE	0.0090	0.0143	0.0154	0.0088	0.0131	0.0133
500	Estimates	1.1595	2.3834	1.5002	1.1584	2.3837	1.5026
	Biases	-0.0405	0.0834	0.0002	-0.0416	0.0837	0.0026
	MSE	0.0042	0.0109	0.0061	0.0042	0.0101	0.0050
1000	Estimates	1.1622	2.3785	1.5010	1.1609	2.3801	1.5017
	Biases	-0.0378	0.0785	0.0010	-0.0391	0.0801	0.0017
	MSE	0.0027	0.0088	0.0031	0.0027	0.0088	0.0026
2500	Estimates	1.1663	2.3692	1.5009	1.1628	2.3760	1.5012
	Biases	-0.0337	0.0692	0.0009	-0.0372	0.0760	0.0012
	MSE	0.0018	0.0064	0.0013	0.0019	0.0077	0.0010

Table **1** shows that both MLE and MPS estimators improve with larger sample sizes, but MLE consistently yields smaller bias and mean squared error (MSE), confirming its higher efficiency.

Table **2** indicates that WLSE performs slightly better than OLSE, particularly for larger samples, yet both remain less efficient than MLE and MPS in terms of MSE reduction.

Table **3** provides a comparative overview showing that MLE consistently achieves the lowest cumulative MSE across all sample sizes, followed by MPS, WLSE, and OLSE.

Table **4** confirms MLE as the best-performing estimator with the lowest overall MSE rank score (12.5), followed by MPS (16), WLSE (20), and OLSE (30). The findings from Tables **1–4** indicate that the maximum likelihood method provides the most accurate and consistent parameter estimates. This advantage arises because it incorporates all sample information and tends to achieve minimum variance as the sample size grows. In comparison, least-squares approaches depend more on empirical distributions, which can increase variability, particularly in small samples. Consequently, MLE is the most dependable option for estimating parameters of the proposed model and is well suited for applications involving medical and survival data.

6. APPLICATIONS TO MEDICAL DATASETS

This section illustrates the application of the TIIHLTL-LLog distribution in the field of medicine. To practically demonstrate its utility, two cancer datasets were explored.Rival distributions such as Inverse Lomax Log-logistic (ILLLog) distribution by [55], Kumaraswamy-Log-logistic (KMLLog) distribution by [56], Odd Exponential Log-Logistic (OELLog) distribution by [57], Inverse Burr Log-Logistic (IBLLog) distribution by [58], and the classical Log-logistic (LLog) distribution are also included for comparison utilizing log-likelihood, the AIC, and the BIC under MLE method. The analysis of the two cancer datasets demonstrates that the proposed TIIHLTLLLog model

Table 3: Summary of the Simulation Studies among the Four Methods of Estimation

Method	N	MLE			$\sum Ranks$
		γ =1.2	θ =2.3	τ =1.5	
MLE	10	0.1333^{1}	0.1777^{4}	1.4068^{4}	9^{3}
MPS	10	0.1340^{2}	0.1164^{1}	0.4569^{1}	4^{1}
OLS	10	0.1691^{4}	0.1542^{3}	0.8176^{3}	10^{4}
WLSE	10	0.1455^{3}	0.1203^{2}	0.6112^{2}	7^{2}
MLE	30	0.0463^{1}	0.0375^{1}	0.1163^{2}	4^{1}
MPS	30	0.0481^{2}	0.0414^{2}	0.0888^{1}	5^{2}
OLS	30	0.0541^{4}	0.0519^{4}	0.1432^{4}	12^{4}
WLSE	30	0.0512^{3}	0.0492^{3}	0.1203^{3}	9^{3}
MLE	70	0.0225^{1}	0.0183^{1}	0.0421^{2}	4^{1}
MPS	70	0.02404^{3}	0.0216^{3}	0.0352^{1}	$7^{2.5}$
OLS	70	0.0243^{4}	0.0229^{4}	0.0487^{4}	12^{4}
WLSE	70	0.0232^{2}	0.0191^{2}	0.0427^{3}	$7^{2.5}$
MLE	100	0.0162^{2}	0.0135^{1}	0.0245^{2}	5^{1}
MPS	100	0.0178^{4}	0.0182^{3}	0.0239^{1}	8^{3}
OLS	100	0.0171^{3}	0.0199^{4}	0.0319^{4}	11^{4}
WLSE	100	0.0161^{1}	0.0161^{2}	0.0277^{3}	6^{2}
MLE	200	0.0085^{1}	0.0113^{1}	0.0120^{2}	4^{1}
MPS	200	0.0092^{4}	0.0138^{3}	0.0118^{1}	8^{3}
LSE	200	0.0090^{3}	0.0143^{4}	0.0154^{4}	11^{4}
WLSE	200	0.0088^{2}	0.0131^{2}	0.0133^{3}	7^{2}
MLE	500	0.0040^{1}	0.0089^{1}	0.0044^{1}	3^{1}
MPS	500	0.0043^{4}	0.0097^{2}	0.0045^{2}	8^{2}
LSE	500	$0.0042^{2.5}$	0.0109^{4}	0.0061^{4}	10.5^{4}
WLSE	500	$0.0042^{2.5}$	0.0101^{3}	0.0050^{3}	8.5^{3}
MLE	1000	0.0027^{3}	0.0077^{1}	0.0020^{1}	$5^{1.5}$
MPS	1000	0.0026^{1}	0.0079^{2}	0.0022^{2}	$5^{1.5}$
LSE	1000	0.0027^{3}	$0.0088^{3.5}$	0.0031^{4}	10.5^{4}
WLSE	1000	0.0027^{3}	$0.0088^{3.5}$	0.0026^{3}	9.5^{3}
MLE	2500	0.0020^{4}	0.0071^{3}	$0.0009^{1.5}$	8.5^{3}
MPS	2500	0.0017^{1}	0.0067^{2}	$0.0009^{1.5}$	4.5^{1}
LSE	2500	0.0018^{2}	0.0064^{1}	0.0013^{4}	7^{2}
WLSE	2500	0.0019^{3}	0.0077^{4}	0.0010^{3}	10^{4}

Table 4: Summary of Overall Ranking Based on MSE of the Four Methods of Estimation

Methods	Overall MSE rank score	MSE final rank
MLE	12.5	1
MPS	16	2
LSE	30	4
WLSE	20	3

provides the best statistical fit, as indicated by the lowest AIC, BIC. Beyond numerical superiority, this result also carries practical medical significance. The superior fit implies that the TIIHLTLLLog distribution more accurately represents the underlying survival behavior of cancer patients. In particular, the model's ability to capture diverse hazard rate shapes such as increasing, decreasing, and bathtub patterns, enables a more realistic description of disease progression and patient risk over time. For instance, a bathtub-shaped hazard may reflect an early postoperative risk that declines with treatment and rises again due to long-term recurrence, while an increasing hazard suggests cumulative risk of mortality as cancer advances. Therefore, the flexible hazard structure of the TIIHLTLLLog model provides oncologists and medical researchers with improved insights into the dynamics of survival and relapse, supporting more accurate prognostic assessment and treatment planning.

Dataset I

This data set contains thirty-three observations (patients), and the data is obtained as the survival times of thirty-three patients suffering from Leukaemia. These data were studied by [59]. It is provided as:

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3,4, 2, 3, 8, 4, 3, 30, 4, 43.

The descriptive statistics for Dataset I include a minimum value of 1.00, a maximum value of 156.00, a mean of 40.88, a first quartile value of 4.00, a third quartile value of 65.00, a standard deviation of 46.70, a skewness of 1.11, and a kurtosis of -0.06.

Figure **4** presents descriptive plots of Dataset I, featuring the histogram, box plot, kernel density, and violin plots.

Table **5** provides the results of the MLEs, log-likelihood, AIC, and BIC of the TIIHLTL-LLog model and competing models for Dataset I. The result in Table **5** indicates that the TIIHLTL-LLog model proved to be the best fit among its contemporary models. Figure **5** supports the results of Table **5**.

Table **5** demonstrates that the TIIHLTLLLog model achieves the lowest AIC (316.25) and BIC (320.74),

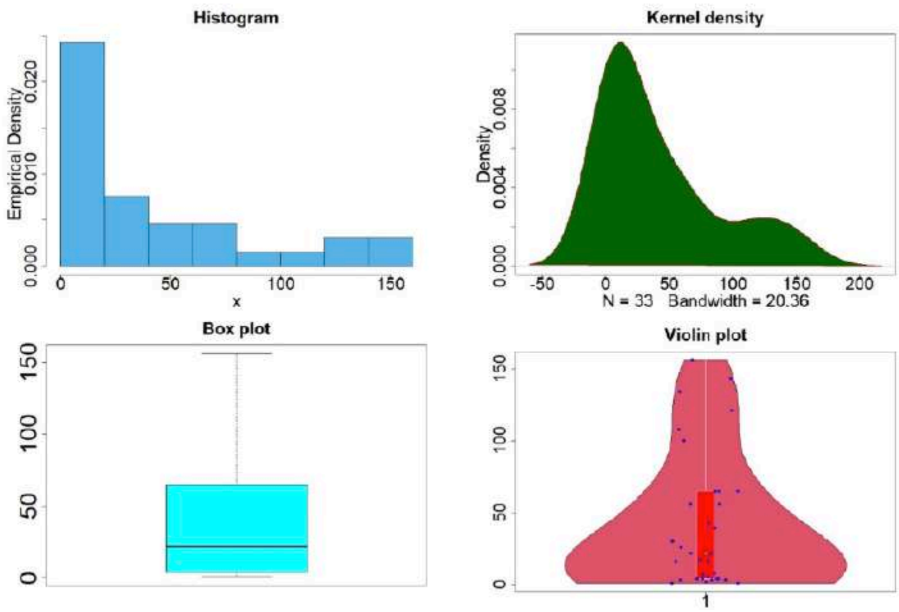

Figure 4: Descriptive plots of Dataset I.

Table 5: MLEs and Goodness-of-Fit for Dataset I

Model	Estimate		Log-likelihood	AIC	BIC
TIIHLTL-LLog	γ	2.6092	-155.1232	316.2464	320.7359
	θ	6.0103			
	δ	0.3969			
ILLLog	α	0.7014	-155.9666	317.9332	322.4227
	θ	61.1502			
	β	0.0704			
KMLLog	Z	0.4369	-160.5384	329.0768	335.0628
	θ	1.7946			
	β	0.0580			
	γ	7.9730			
OELLog	λ	2.3651	-163.5868	333.1736	337.6631
	σ	111.6676			
	θ	0.7765			
IBLLog	θ	0.0228	-157.187	320.374	324.8635
	β	5.0160			
	γ	2.1826			
LLog	β	0.2734	-172.3234	346.6468	348.1433

outperforming all competing models and indicating superior fit for the leukemia survival data.

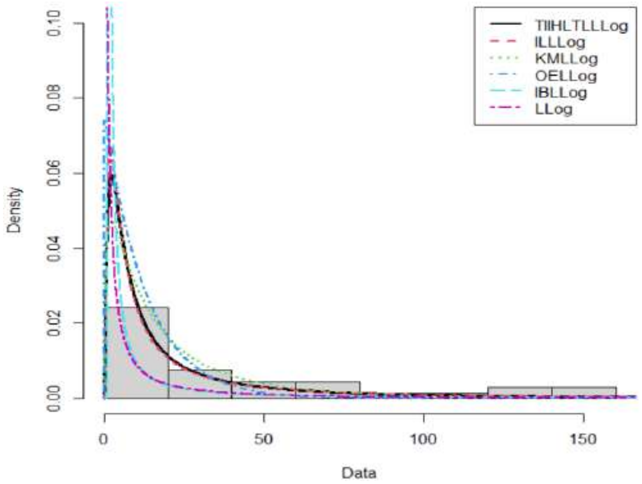

Figure 5: Fitted probability density functions for the TIIHLTLLLog and competing models for Dataset I (Leukemia). The TIIHLTLLLog model aligns most closely with the empirical distribution, indicating superior fit.

The fitted curves in Figure **5** show that the TIIHLTLLLog distribution tracks the empirical data more closely than other models, especially in the right tail, confirming its capacity to model heavy-tailed survival data accurately.

Data Set II

This dataset contains 44 observations (patients), The data is based on the survival rates of patients diagnosed with head and neck cancer who were treated with a combination of radiotherapy and chemotherapy. The dataset was reported by [60]. It is given by:

12.20,23.56,23.74,25.87,31.98,37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84 92, 94, 110, 112, 119,127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 469, 519, 633, 725, 817, 1776.

Descriptive statistics for Dataset II reveal a minimum value of 12.20, a maximum of 1776.00, a mean of 223.48, a first quartile at 67.21, a third quartile at 219.00, a standard deviation of 305.43, a skewness of 3.27, and a kurtosis of 12.82. In Figure **6**, descriptive plots for Dataset II are illustrated, showcasing the histogram, box plot, kernel density, and violin plots.

Table **6** displays the MLEs, log-likelihood, AIC, and BIC for the TIIHLTL-LLog model and its competitors applied to Dataset II. The outcomes in Table **6** underscore that the TIIHLTL-LLog model exhibits the best fit among all considered models. Figure **7** further confirms the findings presented in Table **6**.

Table **6** shows that the TIIHLTLLLog model provides the best fit for the head and neck cancer dataset, with the lowest AIC (545.14) and BIC (550.50) among all compared models.

The fitted curves in Figure **5** show that the TIIHLTLLLog distribution tracks the empirical data more closely than other models, especially in the right tail, confirming its capacity to model heavy-tailed survival data accurately.

Table 6: MLEs and Goodness-of-Fit for Dataset II

Model	Estimate		Log-likelihood	AIC	BIC
TIHLTL-LLog	γ θ δ	14.01908.7115 0.4792	-269.5724	545.1448	550.4974
ILLLog	α θ β	1.0817 10.3417 11.4535	-278.9398	563.8796	569.2322
KMLLog	ζ θ β γ	9.775 10.2780 0.0626 7.2529	-294.3342	596.6684	603.8052
OELLog	λ σ θ	11.6930 15.81810.9406	-281.8417	569.6834	575.036
IBLLog	θ β γ	21.7579 1.2076 0.1164	-277.7041	561.4082	566.7608
LLog	β	0.4847	-285.8111	573.6222	575.4064

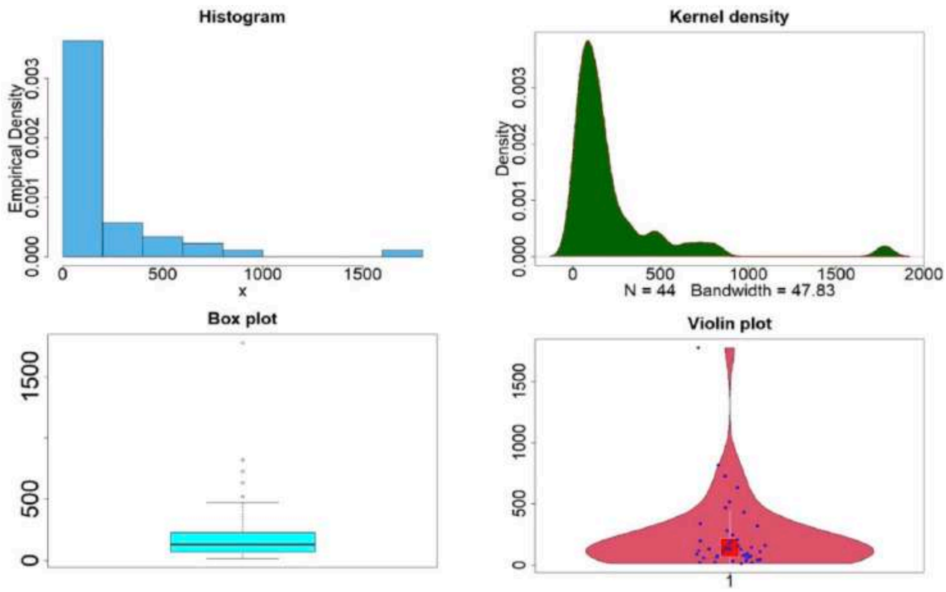

Figure 6: Descriptive plots of Dataset II.

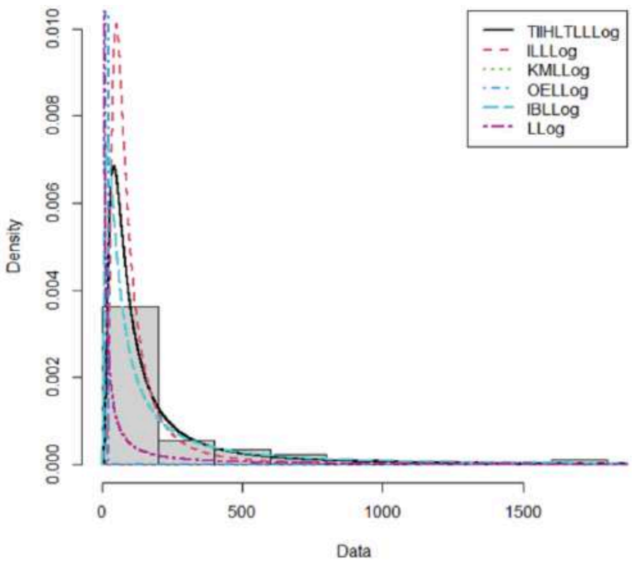

Figure 7: Fitted probability density functions for the TIIHLTLLLog and competing models for Dataset II (survival times). The TIIHLTLLLog model aligns most closely with the empirical distribution, indicating superior fit.

7. CONCLUDING REMARKS

This study proposed the Type II Half Logistic Topp–Leone–G (TIIHLTL–G) family, which integrates the strengths of the Type II Half Logistic–G and Topp–Leone–G models to improve flexibility in modeling skewed, heavy-tailed, and complex hazard patterns. A specific sub-model, the TIIHLTLLLog distribution, was applied to two cancer datasets. For Dataset I, it achieved an AIC of 316.25 and BIC of 320.74, lower than all competing models (e.g., LLog: AIC = 346.65, BIC = 348.14). Similarly, for Dataset II, the model recorded an AIC of 545.14 and BIC of 550.50, outperforming alternatives such as the OELLog (AIC = 569.68, BIC = 575.04). Simulation findings also confirmed that the maximum likelihood estimator (MLE) provided the most accurate and stable parameter estimates with minimal MSE. The proposed model's ability to capture multiple hazard rate shapes—including increasing, decreasing, and bathtub forms, makes it a versatile tool for survival and reliability data analysis. Future research may extend this framework to regression, multivariate modeling, and control chart applications.

DATA AVAILABILITY

All datasets used are included in the paper.

CONFLICTS OF INTEREST

The authors declared no conflicts of interest.

NOMENCLATURE AND NOTATION

To enhance clarity and reduce redundancy, this section consolidates all symbols, parameters, and abbreviations used throughout the paper.

Symbol	**Description**
$T-X$	Transformed transformer
$T \in [a,b]$	T is a random variable bounded with a and b
$-\infty$	Minus infinity
∞	Infinity
$W\left[G(x)\right]$	Cdf
$F_{TIIHL-G}(x;\gamma,\xi)$	Cdf of the TIHL distribution family
$f_{TIIHL-G}(x;\gamma,\xi)$	Pdf of the TIHL distribution family

γ	Parameter (Shape) of the TIHL distribution family
$K(x;\xi)$	Cdf of the parent distribution
$k(x;\xi)$	Pdf of the parent distribution
ξ	Vector parameter
$F_{TL-G}(x;\theta,\tau)$	Cdf of the TL distribution family
$f_{TL-G}(x;\theta,\tau)$	Pdf of the TL distribution family
$H(x;\tau)$	Cdf of the parent distribution
$h(x;\tau)$	Pdf of the parent distribution
θ	Parameter (shape) of the TL distribution family
τ	Parent distribution vector parameter(s)
$F_{TIIHLTL-G}(x;\gamma,\theta,\tau)$	Cdf of the TIHLTL distribution family
$f_{TIIHLTL-G}(x;\gamma,\theta,\tau)$	Pdf of the TIHLTL distribution family
δ	Differentiation
$\frac{o}{\delta x}$	Differentiation of a function with respect to x
$\delta_{r,s}$	PWM
$\upsilon_r^{'}$	Moments
$M_x(t)$	MGF
$R(x)$	Reliability function
$F(x)$	Cdf
$f(x)$	Pdf
$R(x;\gamma,\theta,\tau)$	Reliability function of the TIHLTL distribution family
$H(x;\gamma,\theta,\tau)$	Hazard function of the TIHLTL distribution family
$f_{r,n}(x;\gamma,\theta,\tau)$	Order statistics of the TIHLTL distribution family
$f_{n,n}(x;\gamma,\theta,\tau)$	Maximum order statistics of the TIHLTL distribution family
$f_{1,n}(x;\gamma,\theta,\tau)$	Minimum order statistics of the TIHLTL distribution family
$Q(x)$	Quantile function
$H^{-1}(x;\tau)$	Inverted cdf
$J(x;\tau)$	Cdf of LLog distribution
$j(x;\tau)$	Pdf of LLog distribution
τ	parameter (Scale) of LLog distribution

Abbreviations	**Meaning**
TIIHLTL-G	Type II Half logistic Topp-leone G family
TIIHLTLLLog	Type II Half logistic Topp-leone Log-logistic model
TIIHL-G	Type II Half logistic-G family
TIIEtHL-G	Type II Exponentiated Half-Logistic-G family
TL-G	Topp-leone-G family
HL	Half Logistic
TL	Topp-leone
OBP	Odd Beta Prime
Fr	Frechet
G	Generalized
CDF	Cumulative distribution function
PDF	Probability density function
AIC	Akaike information criterion
BIC	Bayesian information criterion
MSE	Mean square error

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