The Area-Biased Burhan Distribution: Structural Properties, Parametric Inference and Application to Medical Data

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Abstract: Statistical modeling of survival and lifetime phenomena across clinical research, epidemiology, and healthcare analytics frequently encounters complex data structures influenced by ascertainment bias and non-random sampling. In medical applications such as analyzing patient survival times, disease recurrence, or treatment response rates, the weighted distributions offer a mathematically robust mechanism to adjust baseline probability density functions, introducing additional parametric flexibility to capture non-monotonic hazard rates and asymmetric tail behavior. Motivated by these advantages, this study proposes the area-biased Burhan distribution by applying the weighting technique to the classical Burhan distribution. We present a comprehensive theoretical investigation of the proposed model, establishing its basic probability functions, reliability measures, and key statistical properties, including moments, order statistics, and entropy measures. Unknown parameters are estimated using the method of maximum likelihood, and their finite-sample performance is validated through a Monte Carlo simulation study. Finally, the practical utility, empirical adaptability, and superior fitting performance of the area-biased Burhan distribution are demonstrated using real-world lifetime data, highlighting its efficacy as a flexible alternative to traditional lifetime distributions in medical and allied sciences.

Keywords: Area-biased Burhan distribution, Weighted distributions, Reliability measures, Maximum likelihood estimation, Monte Carlo simulation, Lifetime data modeling.

1. INTRODUCTION

In modern medical science and clinical research, probability distributions serve as vital analytical tool for modeling critical health phenomena, such as disease incubation periods, patient survival times, treatment response rates, and epidemic spreading dynamics. Accurate statistical modeling enables epidemiologists, medical statisticians, and healthcare practitioners to extract reliable insights from complex biological datasets where outcome variables often exhibit high variability, heavy tails, or non-standard hazard rates. Beyond clinical research, these statistical tools extend seamlessly across diverse quantitative domains, including reliability engineering, actuary, ecology, finance, and branching processes. However, standard probability models frequently assume pristine experimental conditions, whereas observational health and biological data are typically subject to selection mechanisms and unequal sampling probabilities. To overcome these real-world data collection challenges, weighted distribution frameworks have emerged as an indispensable modeling alternative.

The theoretical foundation for weighted probability models originated with Fisher [1] to correct for ascertainment bias, followed by Rao [2], who systematically formalized the underlying methodology for analyzing datasets collected under non-random, non-replicated, or biased sampling protocols. Rather than assuming equal likelihood across all outcomes, the weighted transformation mechanism modifies a baseline probability density function by introducing an appropriate weight function. This structural adaptation accounts for unequal sampling probabilities, yielding a more precise mathematical description of the observed data-generating process. A distinct benefit of applying weighted transformations in distribution theory is the inclusion of supplementary parameters. Expanding the parameter space improves structural versatility, enabling classical distributions to capture intricate skewness, kurtosis, and varied hazard rate behavior.

Driven by these theoretical advantages, numerous authors have investigated novel weighted probability models and demonstrated their empirical utility across diverse disciplines. Survival data analysis saw significant advancement when Ghitany *et al*. [3] investigated the statistical properties and fitting capability of a two-parameter weighted Lindley model. Expanding on classical baseline distributions, Jain *et al*. [4] formulated the weighted gamma family, while Dey *et al*. [5] derived the weighted exponential distribution alongside various parameter estimation strategies. To extend the flexibility of Rayleigh-based models, Ajami and Jahanshahi [6] constructed the weighted Rayleigh distribution and detailed its estimation procedures. Subsequent research focused on specialized generalizations: Subramanian and Rather [7] established the mathematical foundation for the weighted exponentiated Mukherjee–Islam distribution, and Rather *et al*. [8] analyzed a size-biased variant of

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the Ailamujia distribution tailored for engineering and clinical studies. In the domain of heavy-tailed data modeling, Para and Jan [9] developed the weighted Pareto type-II model and demonstrated its effectiveness in healthcare analytics. Rather and Ozel [10] proposed the weighted power Lindley model, highlighting its superior empirical adaptability compared to other well-known models. Expanding on weighted transformations, Qayoom and Rather [11] formulated the weighted transmuted Mukherjee–Islam distribution, and Rather *et al*. [12] investigated system reliability optimization and structural properties using the weighted Erlang-truncated exponential distribution. Length-biased modeling was further explored by Qayoom and Rather [13] through a comprehensive study of the length-biased transmuted Mukherjee–Islam distribution. Parallel to developments in weighted transformation approach based models, researchers have explored a variety of alternative modeling mechanisms—such as exponentiated, probability-generating, and generalized transformations to enhance baseline distribution flexibility. For instance, Rather and Subramanian [14] proposed a novel exponentiated distribution for engineering applications, while Singh *et al*. [15] analyzed linear combinations of order statistics for the exponentiated Nadarajah–Haghighi distribution. In system reliability and lifetime modeling, Qayoom *et al*. [16] introduced a new class of Lindley distribution, detailing its structural properties, parameter estimation via simulation, and practical applications in real-life data modeling. Expanding on functional transformation families, Ahmad *et al*. [17] introduced a novel class of probability-generating distributions, and Bhat *et al*. [18] developed the weighted generalized interval cumulative residual entropy measure to capture complex information-theoretic properties. In epidemiological and clinical research, Qayoom *et al*. [19, 20] formulated flexible extensions of the Rayleigh and power Lindley distributions to analyze COVID-19 medical survival data, whereas Rather *et al*. [21] constructed truncated life testing mechanisms under the Lomax distribution for quality assurance. More recently, Qayoom *et al*. [22] evaluated the structural attributes of the DUS exponential model, and Qayoom and Rather [23] extended the Quasi-XLindley distribution for lifetime analytics. Motivated by these advantages, this study proposes the area-biased Burhan distribution by applying the weighting technique to the Burhan distribution. The proposed model extends the flexibility of the baseline distribution, and its statistical properties, reliability measures, parameter estimation, and simulation performance are investigated.

The remainder of this paper is organized as follows: Section 2 is the overview of the baseline Burhan distribution. Section 3 introduces the proposed area-biased Burhan distribution and presents its basic functions with graphical illustrations. Section 4 discusses the reliability measures of the distribution. Section 5 derives its important statistical properties. Section 6 presents the maximum likelihood estimation procedure, while Section 7 investigates the order statistics. Section 8 examines the entropy measures. Section 9 reports the Monte Carlo simulation study, whereas Section 10 illustrates the application of the proposed distribution using a real dataset. Finally, Section 11 concludes the paper.

2. BURHAN DISTRIBUTION

Consider a non-negative random variable X following Burhan distribution [24], then the probability density function (PDF) of Burhan distribution is given by.

$$f(x;\alpha,\beta,\theta)=\frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\left(\alpha+\frac{x^{\beta}}{\beta}\right)e^{-\theta x};x>0,\alpha,\beta,\theta>0 \quad (2.1)$$

And the corresponding cumulative distribution function (CDF) of Burhan distribution is

$$F(x;\alpha,\beta,\theta)=\frac{\alpha\beta\theta^{\beta}\gamma(1,\theta x)+\gamma(\beta+1,\theta x)}{\beta(\alpha\theta^{\beta}+\Gamma(\beta))} \quad (2.2)$$

Where α,β,θ are the parameters associated with Burhan distribution such that θ and β are the shape parameters and α is the scale parameter and also $\gamma(\beta+1,\theta x)$ is incomplete gamma function which is defined as

$$\gamma(\beta+1,\theta x)=\int_0^{\theta x}u^{(\beta+1)-1}\,e^{-u}du.$$

Let Y be a non-negative continuous random variable with PDF p(y;τ). Suppose observation of Y are recorded according to weighted function $p_w(\mathrm{y})\geq 0$, which reflect how likely a given value of Y is observed, recorded, or sampled. Then, the weighted distribution has PDF:

$$p_w(\mathrm{y};\tau)=\frac{p_w(\mathrm{y})\mathrm{p}(\mathrm{y};\tau)}{E(\,p_w(\mathrm{y}))} \quad (2.3)$$

Where, τ is the parameter vector associated with a random variable Y and the denominator is the normalizing constant computed as:

$$E(\,p_w(\mathrm{y}))=\int_0^{\infty}p_w(\mathrm{y})\mathrm{p}(\mathrm{y};\tau)\,dy<\infty.$$

Let us consider the weighted function as $p_w(\mathrm{y})=y^k$ where k is the weighted technique exponent, and further assume that Y is following Burhan distribution with PDF and CDF given above in equation (2.1) and equation (2.2) respectively. Therefore, the PDF of weighted Burhan distribution studied by [25] using transformation technique mentioned above in equation (2.3), is given by:

$$f_w(y;\alpha,\beta,\theta,k) = \frac{\theta^{\beta+k+1} y^k (\alpha\beta + y^\beta) e^{-\theta y}}{\alpha\beta\theta^\beta \Gamma(k+1) + \Gamma(\beta+k+1)}; y > 0, \alpha, \beta, \theta > 0 \quad (2.4)$$

And the corresponding CDF of weighted Burhan distribution is derived as

$$F_w(y;\alpha,\beta,\theta,k) = \frac{\theta^\beta \alpha\beta\gamma(k+1,\theta y) + \gamma(\beta+k+1,\theta y)}{\alpha\beta\theta^\beta \Gamma(k+1) + \Gamma(\beta+k+1)}; y > 0, \alpha, \beta, \theta > 0 \quad (2.5)$$

3. AREA-BIASED BURHAN DISTRIBUTION

In this paper, we develop and study the area-biased Burhan distribution (ABBD), a special case of weighted Burhan distribution, corresponding to k=2 in weighted function $p_w(\mathrm{q}) = y^k$. The length- biased extension of Burhan distribution was studied by [26]. Here, we derive the PDF, CDF, and study other statistical properties, including moments, entropy measures, and parameter estimation of the ABBD.

By choosing the weighted function as $p_w(\mathrm{q}) = y^2$, the PDF of ABBD using the same technique in (2.3) is given as,

$$f_{aw}(y;\alpha,\beta,\theta) = \frac{y^2 \theta^{\beta+3} (\alpha\beta + y^\beta) e^{-\theta y}}{2\alpha\beta\theta^\beta + \Gamma(\beta+3)} \quad (3.1)$$

The behavior of the PDF of ABBD for different parameter values is illustrated in Figures **1** and **2** below:

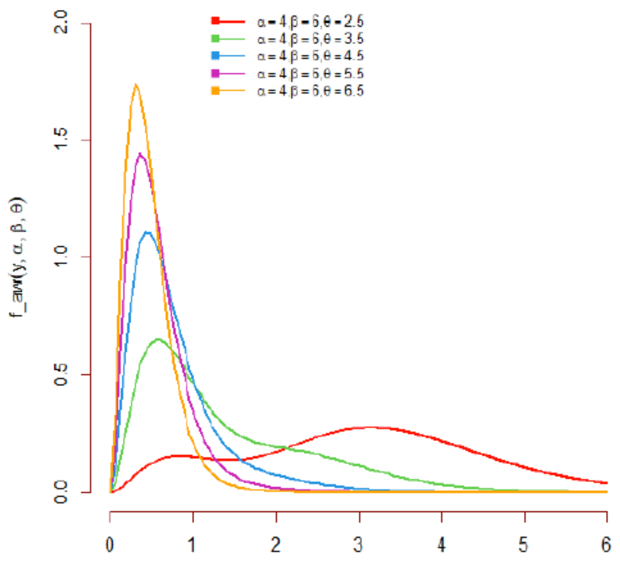

Figure 1: PDF plot of ABBD for different parameters values.

Similarly, the CDF corresponding to ABBD is given by:

$$F_{aw}(y;\alpha,\beta,\theta) = \frac{\alpha\beta\theta^\beta \gamma(3,\theta y) + \gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^\beta + \Gamma(\beta+3)} \quad (3.2)$$

The graphical representation of the CDF of ABBD for different parameter values is illustrated in Figures **3** and **4** below:

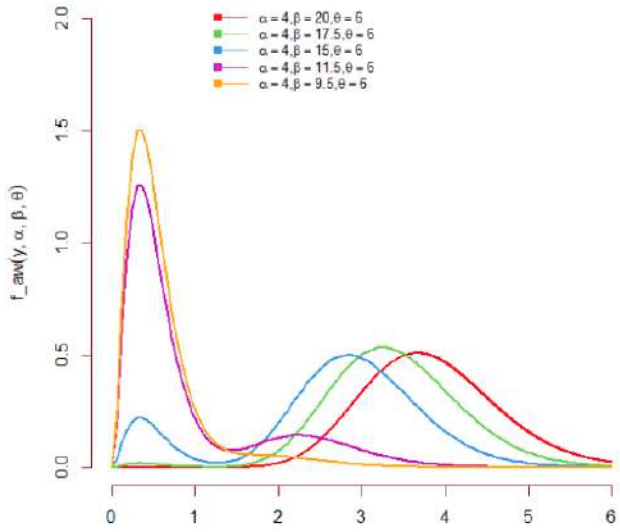

Figure 2: PDF plot of ABBD for different parameters values.

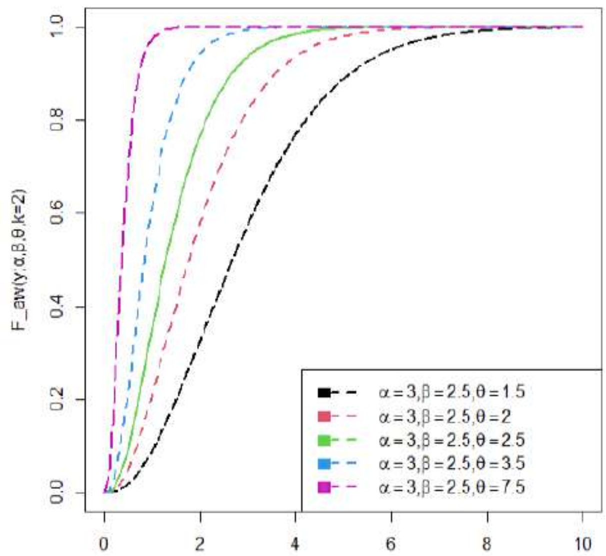

Figure 3: CDF plot of ABBD for different parameter values.

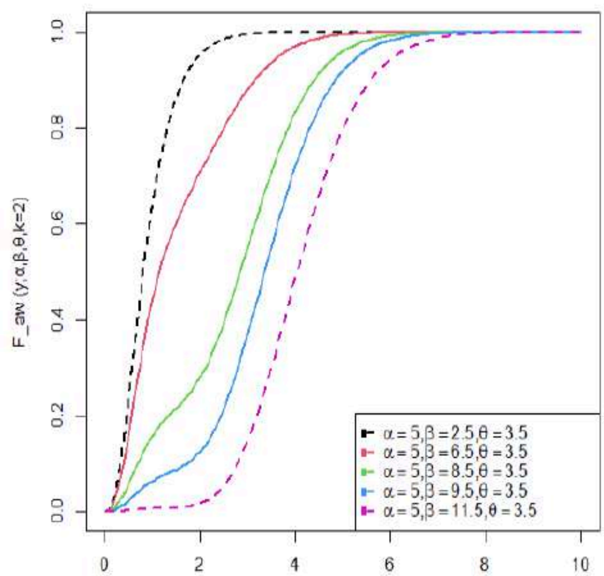

Figure 4: CDF plot of ABBD for different parameter values.

4. RELIABILITY ANALYSIS

Reliability analysis is that branch of statistical study concerned with evaluating and quantifying the dependability of systems, products, or processes over time. It is formally defined as the probability that a given system or component performs its intended function without failure over a stated period, under specified operating conditions. The scope of reliability analysis spans a broad range of disciplines, including engineering, medical and biological sciences, ecology, and industrial systems, among several others, where the assessment of failure behavior and operational lifespan is of practical importance.

4.1. Survival Function

Survival function is one of the fundamental tools used in reliability analysis to describe the probability that a system, component, or unit continues to function beyond a given point in time. It is denoted by S(y) and is mathematically defined as

$$S(y) = 1 - F(y;\Theta) \tag{4.11}$$

Where; $F(y;\Theta)$ is the cumulative distribution.

The survival function of ABBD by using (4.11) is defined as

$$S_{aw}(y) = 1 - F_{aw}(y;\alpha,\beta,\theta)$$

$$S_{aw}(y) = 1 - \frac{\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}$$

$$S_{aw}(y) = \frac{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)-(\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y))}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)} \tag{4.12}$$

The graphical representation of the Survival function of the ABBD is shown in Figure **5** and **6** below:

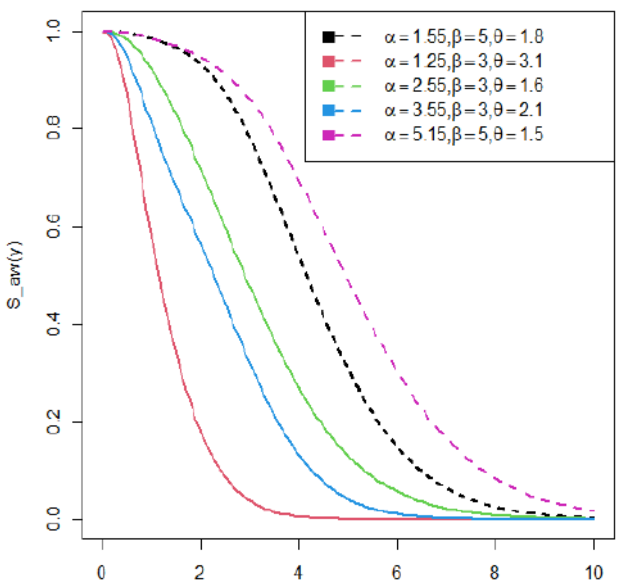

Figure 5: Survival function plot of ABBD for different parameter values.

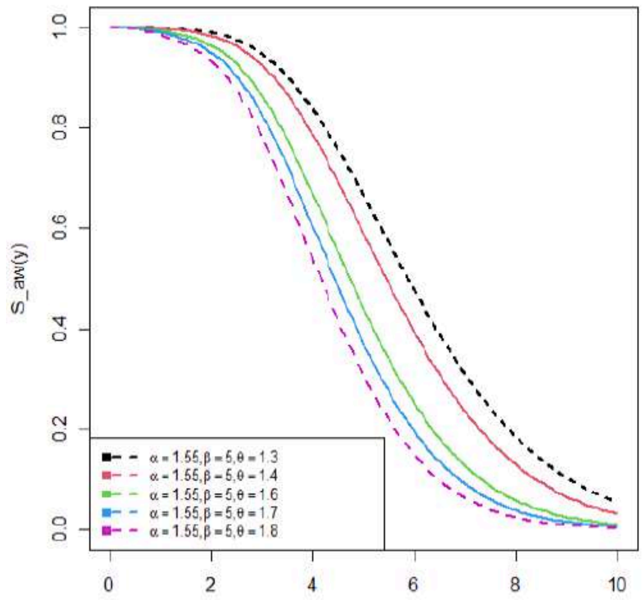

Figure 6: Survival function plot of ABBD for different parameter values.

4.2. Hazard Function

The hazard function, also known as the hazard rate or failure rate function, is another important tool in reliability analysis used to describe the instantaneous risk of failure of a system, component, or unit at a given point in time, given that it has survived up to that time. It is denoted by $h(y)$ and is mathematically defined as

$$h(y) = \frac{f(y)}{S(y)}$$

where $f(y)$ is the PDF and $S(y)$ is the survival function of the lifetime Y. The hazard function represents the conditional probability of failure occurring in the next small interval of time, given that the unit has survived up to time y. Unlike the survival function, the hazard function is not required to be monotonic; its shape, whether increasing, decreasing, constant, or bathtub-shaped, describes the failure pattern of the unit over time and is often used to classify the aging behavior of a distribution.

Hazard function of ABBD denoted by $h_{aw}(y)$ is defined as:

$$h_{aw}(y) = \frac{f_{aw}(y;\alpha,\beta,\theta)}{S_{aw}(y)}$$

Using (3.1) and (4.11) in the above equation we get,

$$h_{aw}(y) = \frac{y^{2}\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{(2\alpha\beta\theta^{\beta} + \Gamma(\beta+3) - (\alpha\beta\theta^{\beta}\gamma(3,\theta y) + \gamma(\beta+3,\theta y))}$$

The behavior of Hazard rate function of the ABBD under different parameter values is displayed in Figures **7** and **8**.

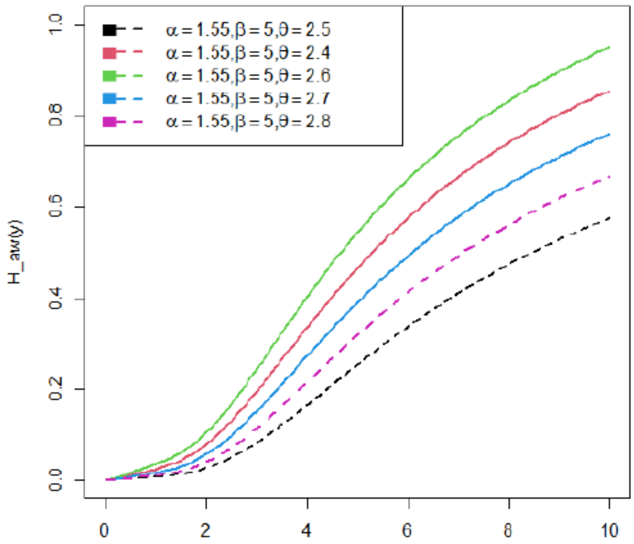

Figure 7: Hazard function plot of ABBD for different values.

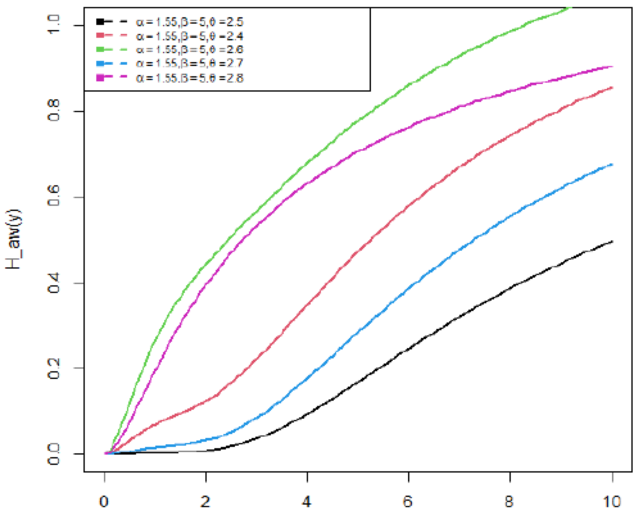

Figure 8: Hazard function plot of ABBD for different values.

5. STRUCTURAL PROPERTIES OF ABBD

5.1. Moment

Moments constitute an essential tool in statistical distribution theory, used to describe and summarize the key characteristics of a probability distribution, such as its central tendency, dispersion, skewness, and kurtosis. The r^{th} raw (or non-central) moment of a continuous random variable X with PDF $f(y)$ is defined as

$$\mu_r' = E(Y^r) = \int_0^\infty y^r f(y)dy$$

where $r = 1,2,3,..$ The first raw moment, $\mu_1' = E(Y)$, gives the mean of the distribution, which represents its average or expected value. Higher order moments provide further information about the shape of the distribution; for instance, the second moment is used in obtaining the variance, while the third and fourth moments are employed in assessing skewness and kurtosis, respectively. These moments are particularly useful in reliability analysis as they allow for the characterization of the lifetime distribution's behavior and facilitate comparison with other competing distributions.

The r^{th} moment about origin of ABBD is given by

$$\mu_r' = \int_0^\infty y^r f_{aw}(y;\alpha,\beta,\theta)dy$$

$$\mu_r' = \int_0^\infty y^r \frac{y^2\theta^{\beta+3}(\alpha\beta+y^\beta)e^{-\theta y}}{2\alpha\beta\theta^\beta+\Gamma(\beta+3)}\, dy$$

Hence,

$$\mu_r' = \frac{\alpha\beta\theta^\beta\Gamma(r+3)+\Gamma(r+\beta+3)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^r} \tag{5.11}$$

Substitute the r =1,2,3,4 in (5.11), we obtain:

$$\mu_1' = \frac{\alpha\beta\theta^\beta\Gamma(4)+\Gamma(\beta+4)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^1} = \frac{6\alpha\beta\theta^\beta+\Gamma(\beta+4)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^1}$$

$$\mu_2' = \frac{\alpha\beta\theta^\beta\Gamma(5)+\Gamma(\beta+5)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^2} = \frac{24\alpha\beta\theta^\beta+\Gamma(\beta+5)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^2}$$

$$\mu_3' = \frac{\alpha\beta\theta^\beta\Gamma(6)+\Gamma(\beta+6)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^3} = \frac{120\alpha\beta\theta^\beta+\Gamma(\beta+6)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^3}$$

$$\mu_4' = \frac{\alpha\beta\theta^\beta\Gamma(6)+\Gamma(\beta+6)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^4} = \frac{720\alpha\beta\theta^\beta+\Gamma(\beta+6)}{(2\alpha\beta\theta^\beta+\Gamma(\beta+3))\theta^4}$$

The variance (σ^2), standard deviation (S.D), Coefficient of variation (CV), skewness (S) and kurtosis(K), of ABBD can be obtained using the following expression:

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

S.D $=\sqrt{\mu_2' - (\mu_1')^2}$

$$C.V = \frac{\sqrt{\mu_2' - (\mu_1')^2}}{\mu_1'}$$

$$S = \frac{\mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3}{(\mu_2' - (\mu_1'))^2}$$

$$K = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1^2\mu_2' - 3\mu_1^4}{\mu_2' - (\mu_1')^2}$$

5.2. Moment Generating Function

The moment generating function of ABBD is given by

$$M_y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f_{aw}(y;\alpha,\beta,\theta,k=2)dy$$

Using Maclaurin series expansion

$$e^{ty} = \sum_{r=0}^{\infty}\frac{(ty)^r}{r!} = \sum_{r=0}^{\infty}\frac{t^r y^r}{r!}$$

By taking the expectation

$$E(e^{ty}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(y^r)$$

And

$$E(y^r) = \mu_r' = \frac{\alpha\beta\theta^{\beta}\Gamma(r+3)+\Gamma(r+\beta+3)}{(2\alpha\beta\theta^{\beta}+\theta^{\beta}\Gamma(\beta+3))\theta^r}$$

Therefore,

$$M_y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\alpha\beta\theta^{\beta}\Gamma(r+3)+\Gamma(r+\beta+3)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3))\theta^r}$$

Similarly, the characteristic function denoted by $Q_y(t)$ of the of ABBD is given as:

$$Q_y(t) = E(e^{ity}) = M_y(it)$$

Therefore,

$$Q_y(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\alpha\beta\theta^{\beta}\Gamma(r+3)+\Gamma(r+\beta+3)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3))\theta^r}$$

The cumulant generating function denoted by $C_y(t)$ associated with ABBD is simply log of moment generating function, that is $\log(M_y(t)))$ and is given as:

$$C_y(t) = \log\left\{\sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\alpha\beta\theta^{\beta}\Gamma(r+3)+\Gamma(r+\beta+3)}{\left(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)\right)\theta^r}\right\}$$

6. ESTIMATION OF PARAMETERS

Parameter estimation is a crucial aspect of statistical distribution theory, concerned with obtaining suitable estimates of the unknown parameters of a distribution based on a set of observed data. Among the various methods available, the method of maximum likelihood is the most widely adopted, owing to its desirable asymptotic properties such as consistency, efficiency, and asymptotic normality.

Given a random sample $Y_1, Y_2, Y_3 \dots Y_n$ drawn from ABBD and assume that $y_1, y_2, y_3 \dots y_n$ be the corresponding sample values. Then the likelihood function of the given sample is:

$$L(\alpha,\beta,\theta) = \prod_{i=1}^{n} f_{aw}(y;\alpha,\beta,\theta)$$

$$L(\alpha,\beta,\theta) = \prod_{i=1}^{n} \frac{{y_i}^2\theta^{\beta+3}(\alpha\beta + {y_i}^{\beta})e^{-\theta y_i}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}$$

$$L(\alpha,\beta,\theta) = \frac{\theta^{n(\beta+3)}}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3))^n} \prod_{i=1}^{n} {y_i}^2 \prod_{i=1}^{n} (\alpha\beta + {y_i}^{\beta})\, e^{-\sum_{i=1}^{n}\theta y_i}$$

Take the logarithm to both sides of the likelihood function,

$$\ell(\alpha,\beta,\theta,k=2) = \log L(\alpha,\beta,\theta,k=2)$$

$$= n(\beta+3)log\theta + 2\sum_{i=1}^{n} log\, y_i + \sum_{i=1}^{n} log\,(\alpha\beta + {y_i}^{\beta}) - \theta\sum_{i=1}^{n} y_i - n(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3))$$

$$\frac{d\ell}{d\alpha} = \sum_{i=1}^{n} \frac{\beta}{(\alpha\beta + {y_i}^{\beta})} - \frac{2n\beta\theta^{\beta}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)} = 0$$

$$\frac{d\ell}{d\theta} = nlog\theta + \sum_{i=1}^{n} \frac{\alpha + {y_i}^{\beta} logy_i}{(\alpha\beta + {y_i}^{\beta})} - \frac{n(2\alpha\theta^{\beta}(1+\beta log\theta + \Gamma(\beta+3)\psi(\beta+3)}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)} = 0$$

$$\frac{d\ell}{d\alpha} = \frac{n(\beta+3)}{\theta} - \sum_{i=1}^{n} y_i - n\frac{2\,\alpha\beta^2\theta^{\beta-1}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)} = 0$$

7. ORDER STATISTICS

Order statistics play a significant role in various areas of statistics, such as estimation of parameters, hypothesis testing, and confidence interval estimation. It usually deals with the study and analysis of data in which the order or rank of the observations is considered.

Let $Y_1, Y_2, Y_3, \dots Y_m$ be a random sample of size m drawn from the ABBD, with corresponding sample values $y_1, y_2, y_3 \dots y_n$. Then the ordered statistics based on the given sample is $Y_{(1)}, Y_{(2)}, Y_{(3)}, \dots Y_{(m)}$ such that $Y_{(1)} \le Y_{(2)} \le Y_{(3)} \le \dots \le Y_{(m)}$

Where $Y_{(1)} = \min(Y_1, Y_2, Y_3, \dots Y_m)$ and $Y_{(m)} = \max(Y_1, Y_2, Y_3, \dots Y_m)$

Therefore, the PDF of the i^{th} order statistic $y_{(i)}$ of the Area-Weighted distribution is given by

$$f_{y_{(i)}}(y) = \frac{m!}{(m-i)!(i-1)!}\left(f_{aw}(y;\alpha,\beta,\theta,)\right)\left(F_{aw}(y;\alpha,\beta,\theta)\right)^{(i-1)}\left(1 - F_{aw}(y;\alpha,\beta,\theta)\right)^{(m-i)} \quad (8.1)$$

$; \forall i = 1,2, \dots m$

Using the PDF and CDF of the area-weighted distribution derived earlier, we substitute into (8.1):

$$f_{y_{(i)}}(y) = \frac{m!}{(m-i)!(i-1)!} \times \frac{y^2\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}$$

$$\times\left(1 - \frac{\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{(m-1)} \left(\frac{\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{(i-1)} \quad (8.2)$$

For $i = 1, m$ in the above equation we obtain the pdf of smallest ordered ($Y_{(1)}$) and highest ordered statistics ($Y_{(m)}$) respectively of ABBD and are express as:

$$f_{y_{(1)}}(y) = m\left(1 - \frac{\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{(m-1)}\left(\frac{y^2\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)$$

and

$$f_{y_{(m)}}(y) = m\left(\frac{\alpha\beta\theta^{\beta}\gamma(3,\theta y)+\gamma(\beta+3,\theta y)}{(2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{(m-1)}\left(\frac{y^2\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)$$

8. ENTROPY

Entropy is a fundamental concept in statistics that provides a quantitative measure of the information content and uncertainty present in a dataset, and is essential in various applied fields such as statistics, communication theory, data analysis, machine learning, and information theory. In this section, we derive two fundamental generalizations of the Shannon entropy [27] measure, namely: Renyi Entropy [28], and Tsalli's Entropy [29] which are discussed as given below.

8.1. Renyi Entropy

The Renyi entropy denoted as $R_n(y)$ of a continuous random variable Y following ABBD is defined as

$$R_n(y) = \frac{1}{(1-\delta)}\log\left(\left(\int_0^{\infty} f_{aw}(y;\alpha,\beta,\theta)\right)^{\delta} dy\right)$$

$$R_n(y) = \frac{1}{(1-\delta)}\log\left(\int_0^{\infty}\left(\frac{y^2\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{\delta} dy\right)$$

Let A = $\frac{\theta^{\beta+3}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}$

Then

$$f_{aw}(y;\alpha,\beta,\theta) = Ay^2(\alpha\beta + y^{\beta})e^{-\theta y}$$

To get $R_n(y)$ in closed form we expand $(\alpha\beta + y^{\beta})^{\delta}$ as a binomial series

$$(\alpha\beta + y^{\beta})^{\delta} = \sum_{k=0}^{n}\binom{\delta}{i}y^{i}(\alpha\beta)^{\delta-i} \tag{9.11}$$

Using (9.11) above

$$\int_0^{\infty}\left(f_{aw}(y;\alpha,\beta,\theta)\right)^{\delta}dy$$

$$= A^{\delta}\sum_{k=0}^{n}\binom{\delta}{i}(\alpha\beta)^{\delta-i}\int_0^{\infty} y^{i\beta}y^{2\delta}e^{-\delta\theta y}dy$$

$$= A^{\delta}\sum_{k=0}^{n}\binom{\delta}{i}(\alpha\beta)^{\delta-i}\int_0^{\infty} y^{i\beta+2\delta}e^{-\delta\theta y}dy$$

Also, $\int_0^{\infty} y^{i\beta+2\delta}e^{-\delta\theta y}dy = \frac{(\Gamma(i\beta+2\delta)}{\delta\theta^{(i\beta+2\delta)}}$

So the Renyi entropy become:

$$R_n(y) = \frac{1}{(1-\delta)} = \left[\delta log\left(\frac{\theta^{\beta+3}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right) + log\left(\sum_{k=0}^{n}\binom{\delta}{i}(\alpha\beta)^{\delta-i}\frac{(\Gamma(i\beta+2\delta)}{\delta\theta^{(i\beta+2\delta)}}\right)\right]$$

8.2. Tsalli's Entropy

Similarly, the Tsalli's entropy of ABBD is given by

$$T_n(y) = \frac{1}{(\delta-1)} = \left[1 - \int_0^{\infty}\left(f_{aw}(y;\alpha,\beta,\theta,k=2)\right)^{\delta}dy\right]$$

$$T_n(y) = \frac{1}{(\delta-1)} = \left[1 - \int_0^{\infty}\left(\frac{y^2\theta^{\beta+3}(\alpha\beta + y^{\beta})e^{-\theta y}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{\delta}dy\right]$$

$$T_n(y) = \frac{1}{(\delta-1)} = \left[1 - \left(\frac{\theta^{\beta+3}}{2\alpha\beta\theta^{\beta}+\Gamma(\beta+3)}\right)^{\delta}\left(\sum_{k=0}^{n}\binom{\delta}{i}(\alpha\beta)^{\delta-i}\frac{(\Gamma(i\beta+2\delta)}{\delta\theta^{(i\beta+2\delta)}}\right)\right]$$

9. SIMULATION

A simulation study was conducted to assess the finite-sample performance of the maximum likelihood estimators for the ABBD. Data were generated using the inverse CDF method, solved numerically since the CDF has no closed form. The samples of size n = 35, 60, 100, 150, 225, 300, and 400 were drawn, and for each size 2000 datasets were simulated and underlying model parameters were estimated via maximum likelihood method. Average bias, MSE, mean relative error, variance, and standard error were computed across replications. As expected, bias and MSE for θ and β decreased steadily as n increased, confirming the estimators' consistency. The parameter α showed higher variability at smaller sample sizes, with a few replications producing estimates far from the true value, a pattern common for shape-type parameters.

The finite-sample performance of the maximum likelihood estimators is summarized in Tables **1**, **2**, **3** and **4**, below where the Bias, Mean Squared Error (MSE), Variance and Mean Relative Error (MRE) are reported for different sample sizes and parameter combinations.

10. APPLICATION

In this section, the proposed ABBD is applied to a real-life dataset to demonstrate its practical usefulness. The data set reported by Gross and Clark [30] consist of relief times in minutes after receiving analgesic for 20 patients. The aim is to investigate how well the proposed distribution fits the data in comparison with

Table 1: Comparison of Estimator Performance Based on Bias, MSE, Variance and MRE of the ABBD when (θ =1.05, α = 0.05, β = 1.25)

Sample Size	Parameter	Bias	MSE	Variance	MRE
30	θ	0.332522	0.193566	0.082995	0.316688
	α	1.222477	21.887336	20.392886	24.449542
	β	1.432498	3.666568	1.614517	1.145998
60	θ	0.233736	0.095097	0.040464	0.222605
	α	0.670289	4.388669	3.939382	13.405780
	β	1.033638	1.908708	0.840300	0.826911
100	θ	0.181753	0.059042	0.026008	0.173098
	α	0.486279	1.045866	0.809399	9.725589
	β	0.828737	1.250540	0.563735	0.662989
150	θ	0.145106	0.036671	0.015615	0.138196
	α	0.383367	0.671200	0.524230	7.667348
	β	0.668124	0.806061	0.359671	0.534499
225	θ	0.116590	0.023902	0.010309	0.111038
	α	0.307071	0.371135	0.276842	6.141420
	β	0.543470	0.540429	0.245069	0.434776
300	θ	0.096787	0.016078	0.006710	0.092178
	α	0.246457	0.205170	0.144429	4.929142
	β	0.446109	0.358618	0.159605	0.356888
400	θ	0.086069	0.012657	0.005249	0.081971
	α	0.223744	0.155091	0.105030	4.474877
	β	0.400154	0.290060	0.129937	0.320123

Table 2: Comparison of Estimator Performance Based on Bias, MSE, Variance and MRE of the ABBD when (θ =1.25, α = 0.10, β = 1.45)

Sample Size	Parameter	Bias	MSE	Variance	MRE
30	θ	0.387342	0.276338	0.126304	0.309873
	α	0.818486	5.147132	4.477213	8.184861
	β	1.467280	3.935575	1.782664	1.011917
60	θ	0.283909	0.139171	0.058567	0.227127
	α	0.527261	2.086515	1.808511	5.272608
	β	1.089010	2.093721	0.907778	0.751041
100	θ	0.205347	0.074067	0.031900	0.164277
	α	0.364468	0.561943	0.429106	3.644681
	β	0.809664	1.158964	0.503408	0.558389
150	θ	0.167896	0.047553	0.019364	0.134317
	α	0.284242	0.236208	0.155414	2.842425
	β	0.659578	0.764237	0.329194	0.454881
225	θ	0.137134	0.032052	0.013246	0.109707
	α	0.238679	0.147460	0.090492	2.386793
	β	0.544928	0.525760	0.228813	0.375812
300	θ	0.115108	0.022549	0.009299	0.092086
	α	0.204138	0.102100	0.060428	2.041383
	β	0.453988	0.367036	0.160931	0.313095
400	θ	0.098912	0.016411	0.006627	0.079130
	α	0.171809	0.065863	0.036345	1.718090
	β	0.386214	0.262974	0.113813	0.266355

Table 3: Comparison of Estimator Performance Based on Bias, MSE, Variance and MRE of the ABBD when (θ =1.20, α = 0.80, β = 2.50)

Sample Size	Parameter	Bias	MSE	Variance	MRE
30	θ	0.308707	0.170495	0.075195	0.257256
	α	5.638727	99.10950	67.31426	7.048409
	β	1.481996	3.756773	1.560461	0.592798
60	θ	0.232863	0.091398	0.037173	0.194052
	α	2.249642	41.00123	35.940338	2.812052
	β	1.142454	2.090568	0.785367	0.456982
100	θ	0.183287	0.053883	0.020289	0.152739
	α	1.290489	5.602657	3.937295	1.613112
	β	0.905414	1.266417	0.446642	0.362165
150	θ	0.155211	0.037855	0.013765	0.129343
	α	1.014476	2.938974	1.909812	1.268096
	β	0.783387	0.939803	0.326108	0.313355
225	θ	0.129017	0.025741	0.009096	0.107514
	α	0.726848	1.120617	0.592309	0.908560
	β	0.654724	0.646828	0.218164	0.261890
300	θ	0.110340	0.018871	0.006696	0.091950
	α	0.624806	0.780429	0.390046	0.781008
	β	0.561197	0.483426	0.168484	0.224479
400	θ	0.099024	0.015070	0.005264	0.082520
	α	0.529928	0.510449	0.229625	0.662410
	β	0.507814	0.386266	0.128391	0.203126

Table 4: Comparison of Estimator Performance Based on Bias, MSE, Variance and MRE of the ABBD when (θ =1.10, α = 0.60, β = 1.80)

Sample Size	Parameter	Bias	MSE	Variance	MRE
30	θ	0.320209	0.186681	0.084147	0.291099
	α	2.867728	111.5230	103.2992	4.779547
	β	1.449858	3.657846	1.555758	0.805477
60	θ	0.231580	0.091121	0.037492	0.210528
	α	1.681525	24.28474	21.45721	2.802542
	β	1.090925	1.967013	0.776896	0.606069
100	θ	0.184130	0.056091	0.022187	0.167391
	α	1.087089	3.643934	2.462172	1.811815
	β	0.904753	1.289373	0.470795	0.502641
150	θ	0.143682	0.034266	0.013621	0.130620
	α	0.871942	2.070533	1.310250	1.453236
	β	0.728158	0.845573	0.315359	0.404532
225	θ	0.120595	0.023183	0.008640	0.109632
	α	0.672825	0.979388	0.526695	1.121375
	β	0.615436	0.582128	0.203367	0.341909
300	θ	0.110668	0.019364	0.007117	0.100608
	α	0.561452	0.595072	0.279844	0.935754
	β	0.561160	0.477750	0.162849	0.311756
400	θ	0.097552	0.014665	0.005149	0.088684
	α	0.499922	0.423719	0.173797	0.833203
	β	0.509255	0.383323	0.123982	0.282920

Table 5: Model Performance Measures Based on Given Data Set

Distribution	Parameters	MLE (S.E)	$-2ln\mathcal{L}$	AIC	BIC	CAIC	HQIC
ABBD	θ	4.9853 (0.6116)	35.7704	41.7705	44.7576	43.2704	42.3536
	α	6.4960 (1.6508)					
	β	0.001 (-)					
BD	θ	4.0993 (0.4002)	37.1457	43.1457	46.1329	44.6457	43.7288
	α	6.9743 (1.2584)					
	β	0.001 (-)					
ETED	β	0.8244 (74.331)	65.6742	69.6742	71.6656	70.3800	70.0629
	θ	1.0173 (159.197)					
WD	α	2.7870 (0.4273)	41.1728	45.1728	47.1643	45.8786	45.5615
	β	2.1300 (0.1820)					
RD	θ	1.4285 (0.1597)	44.9576	46.9576	47.9534	47.1798	47.1520
LD	θ	0.8162 (0.1361)	60.4991	62.4991	63.4948	62.7213	62.6935

some existing lifetime distributions. The data set is given below:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

The parameters of the ABBD are estimated using the maximum likelihood estimation (MLE) method. For comparison, the Burhan Distribution (BD), Erlang Truncated Exponential Distribution (ETED), Weibull Distribution (WD), Rayleigh Distribution (RD), and Lindley Distribution (LD) are also fitted to the same dataset. To determine the most appropriate model, several information criteria, including the Log-Likelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), and Hannan–Quinn Information Criterion (HQIC), are computed. In addition, graphical measure such as the histogram with the fitted probability density functions is used to assess the goodness-of-fit of the competing models. The distribution with the smallest information criterion values and the largest Log-Likelihood is considered to provide the best fit for the data. The information criterion measures are defined as:

$$AIC = -2ln\mathcal{L} + 2\rho$$

$$BIC = -2ln\mathcal{L} + \rho\ln(\mathrm{n})$$

IC = $AIC + \frac{2\rho(\rho+1)}{n-\rho-1}$

HQIC = $-2ln\mathcal{L} + 2\rho\, ln\{ln(n)\}$

Where ρ is the number of parameters in the given distribution and n is the number of data point in the given data set. The graphical plot represented in Figure **9**, illustrate a more accurate representation of the data by ABBD and hence, indicating that the ABBD distribution provides an excellent fit. Moreover, on the basis of information criteria measures, the model with the lowest values of AIC, BIC, CAIC, and HQIC offers the best fit. Therefore, as shown in Table **5**, we conclude that the ABBD, fit the data better than the other competitive distribution including BD, ETED, RD and LD.

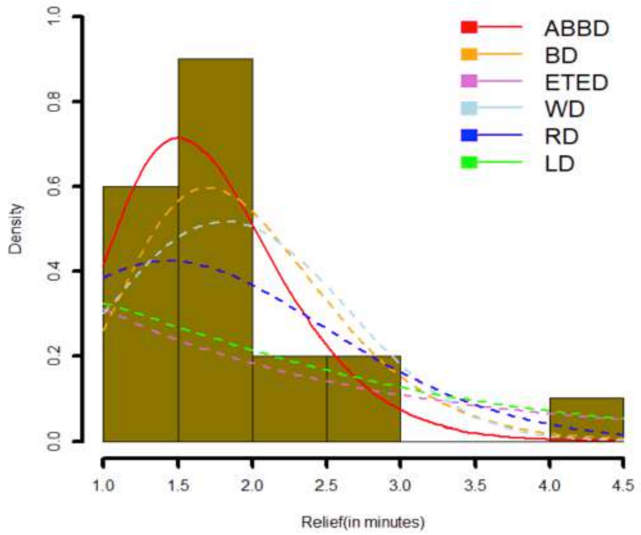

Figure 9: Fitted density curve with histogram of the given data set.

11. CONCLUSION

In this paper, we introduced a novel extension of the Burhan distribution, termed the area-biased Burhan distribution (ABBD), by incorporating the weighting technique. We conducted a detailed theoretical investigation of the proposed ABBD, establishing its explicit mathematical and physical properties. These included closed-form expressions for its probability density and cumulative distribution functions, basic reliability measures, statistical moments, order statistics, and information-theoretic measures such as entropy. To address practical implementation, parameter estimation was executed using the method of maximum likelihood, and its finite-sample behavior was verified via Monte Carlo simulations. The real-world applicability and empirical superiority of the proposed ABBD were demonstrated using a clinical dataset reporting relief times (in minutes) for 20 patients following the administration of an analgesic. Parameter estimates were obtained via maximum likelihood estimation (MLE), and the performance of the ABBD was assessed against the baseline Burhan Distribution (BD), Erlang Truncated Exponential Distribution (ETED), Weibull Distribution (WD), Rayleigh Distribution (RD), and Lindley Distribution (LD). Goodness-of-fit assessments based on key goodness-of-fit metrics specifically the Log-Likelihood, AIC, BIC, CAIC, and HQIC, alongside visual inspection of fitted probability density overlays, confirmed that the ABBD yielded the lowest information criteria values across all competing models.

Overall, the empirical and theoretical findings confirm that the ABBD provides a highly competitive and superior alternative to standard lifetime models in health, medical sciences, and reliability modeling. Future research directions may involve expanding the proposed model under various censoring schemes and exploring its bivariate or regression generalizations.

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