Correlations between the Market Price of Interest Rate Risk and Bond Yields

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Abstract: This paper examines empirical properties of the market price of interest rate risk, focusing on the relation between the price and interest rates. We briefly summarize how the market price of risk is estimated, and introduce the positive slope model to explain our empirical observation. The market price of risk is estimated for the U.S. Treasury market, 1970-2014, using the Hull–White model. We test the correlation between the market price of interest rate risk and bond yields. The results are that the yield change and term spreads are significantly correlated with the market price of risk, but the initial yields are not correlated with that. These results are theoretically interpreted by a mathematical model, and serve as a valuable reference for risk management as well as for study of financial policy.

Keywords: Correlation analysis, Financial policy-making, Hull–White model, Interest rate model, Interest-rate-risk management, Market price of risk, Potential future exposure, Risk premium, Solvency risk, U.S. Treasury yields.

1. INTRODUCTION

Financial risk management has been increasingly seen as important since the global financial crisis of 2008. For example, counterparty credit risk (CCR) management in banks and solvency risk management in life insurance companies have become important tasks in both financial business and the study of finance. One standard risk metric for managing CCR is potential future exposure (PFE). With this, our consideration of the interest rate risk for PFE includes scenarios for the interest rate under a real-world (or historical) measure. Additionally, when assessing solvency risk, interest rate risk is evaluated by nested simulation. In this approach, a first, outer simulation is used to generate financial scenarios and a follow-up, inner simulation is used to estimate future portfolios. Naturally, the outer simulation should be constructed using the real-world measure.

Interest rate models have been successfully used to price derivatives, with a risk-neutral measure assumed in the underlying mathematical formulation. To apply this technique in an interest rate simulator in the context of risk management, we should use the real-world measure, as mentioned above. Doing so creates a “real-world” model in practice. For this, the market price of interest rate risk (hereinafter, market price of risk) is the key parameter needed to realize a real-world model.

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On the topic of estimating the market price of risk, there have been some studies of the calibration of affine term structure models, including Stanton (1997). In these, the market price of risk has been studied in connection with economic states variables, such as short interest rate (see De Jong and Santa-Clara (1999)). In risk management, the chosen interest rate model should make it possible to represent an arbitrage term structure of the yield (or forward rate) curve, and the interest rate scenario should be generated so as to simulate the historical dynamics of the forward rates. These subjects can be handled in a forward rate model, such as the Heath–Jarrow–Morton (HJM) model, Heath *et al*. (1992) or the LIBOR market model introduced by Brace *et al*. (1997), Musiela and Rutkowski (1997), and Jamshidian (1997). They cannot be handled in affine models. Toward the aim of estimating the market price of risk within the HJM framework, Yasuoka (2015a) proposes estimation in the Gaussian HJM model (cf. Yasuoka (2013) in the LIBOR market model.), where it is assumed that the market price of risk is constant during the observation periods. The constancy assumption is considered to be suitable for application to interest rate simulation in the context of risk management, see Yasuoka (2015b), Chapter 7 for details.

The empirical study of the market price of risk is traditionally an important subject in the field of finance, such as the study of risk premia in bond markets, or the study of financial policy making. However, it is better to study this issue by adopting a theoretical perspective to some extent. Taking this point of view, this paper aims to empirically examine the properties of the market price of risk, focusing on the relation between the

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market price of risk and interest rates. These relations serve as a valuable reference for risk management as well as for study of financial policy.

To simplify our investigation, we work with the interest rate model of Hull and White (1990), regarding it as a kind of Gaussian HJM model. Section 2 introduces the method used to estimate the market price of risk in the Gaussian HJM model and in the Hull–White model, summarizing the discussions of Yasuoka (2015a, 2015b). Section 3 empirically investigates the U.S. Treasury market, using data covering 1970 to 2016. The market prices of risk are estimated for sub-periods across the whole period, examining the correlations between the market price of interest rate risk and bond yields. Section 4 summarizes our results, which are theoretically interpreted by a mathematical model to estimate the market price of risk.

2. ESTIMATION METHOD FOR MARKET PRICE OF RISK

In managing interest rate risk, it is practical to work with a forward rate model, such as the HJM model or the LIBOR market model. Although the Hull–White model was originally studied as a short rate model, it is also a special case of the Gaussian HJM model. Because the Hull–White model is simple and practical, it is widely used in financially oriented businesses. The aim of this paper is the study of the market price of risk for the risk management in financial institutions. For these reasons, we work with the Hull–White model in studying the empirical properties of the market price of risk.

This section briefly introduces estimation of the market price of risk, following Yasuoka (2015a, 2015b). Section 2.1 summarizes estimation within the Gaussian HJM model. Applying this method more explicitly, Section 2.3 introduces estimation in the Hull–White model. Additionally, Section 2.4 introduces the "positive slope model" to explain a numerical property of the market price of risk.

2.1. The Gaussian HJM Model

We denote by $f(t,T)$ the instantaneous forward rate (hereinafter, the forward rate) with maturity T prevailing at time $t \le T$. Naturally, the instantaneous spot rate (hereinafter, the spot rate) r is given by $r(t) = f(t,t)$. The dynamics of $f(t,T)$ is represented in the HJM model by

$$df(t,T) = \left(-\sigma(t,T)\cdot \upsilon(t,T) + \sigma(t,T)\cdot \varphi(t)\right)dt + \sigma(t,T)\cdot dW, \tag{2.1}$$

where $\cdot$ denotes the inner product in $\mathbf{R}^d$, and W is a d-dimensional Brownian motion under the real-world measure $\mathbf{P}$. $\varphi(t)$ denotes the vector of the market price of risk such that $\varphi = (\varphi_1,\cdots,\varphi_d)^T$, where the superscript T denotes transposition. Note that $\varphi(t)$ is a stochastic process, see Munk (2011), Chapter 4 or Shreve (2004), Chapter 10. $\sigma(t,T)$ denotes a d-dimensional volatility of $f(t,T)$, and $\upsilon(t,T)$ is defined as

$$\upsilon(t,T) = -\int_t^T \sigma(t,u)du. \tag{2.2}$$

Denoting the price of a T-maturity bond at time t as $B(t,T)$, there is a drift process $\mu(t)$ such that

$$\frac{dB(t,T)}{B(t,T)} = \mu(t,T)dt + \upsilon(t,T)\cdot dW, \tag{2.3}$$

and it is known that μ satisfies the following relation:

$$\mu(t,T) - r(t) = \upsilon(t,T)\cdot\varphi(t). \tag{2.4}$$

Helpfully, (2.4) coincides with a well-known definition of the market price of risk for the bonds market.

When we consider a one-dimensional case of (2.4), we have

$$\varphi(t) = \frac{\mu(t,T) - r(t)}{\upsilon(t,T)}. \tag{2.5}$$

Because $\mu(t,T)$ represents the expected return of a T-maturity bond, the numerator of the right-hand term is the excess return over $r(t)$ for that T-maturity bond. In this context, the traditional interpretation is that the market price of risk represents a kind of risk premium for the bond. Since $\upsilon(t,T)$ is a bond-price volatility from (2.3), the market price of risk has been characterized by the Sharpe ratio.

2.2. Estimation of the Market Price of Risk

The HJM model is Gaussian if $\sigma(t,T)$ is a deterministic function in t and T. In what follows, an $\mathbf{R}^d$-valued volatility $\sigma(t,T)$ is assumed to be always deterministic and continuous in t and T. This means that $\upsilon(t,T)$ is deterministic and continuous.

We denote by $x = T - t$ the time length to a maturity T from t. For an integer $n \ge d$, let $x_1,\cdots,x_n$ be a sequence of time length, where $x_i < x_{i+1}$ for all $i < d$. Let a time interval $\Delta t > 0$ be fixed, and let $\{t_k\}_{k=1,\cdots,J+1}$ be a sequence of observation times with $t_1 = 0$ and $t_{k+1} - t_k = \Delta t$, where $J+1$ is the number of observation times. In the practice, we observe the instantaneous forward rate $F(t_k, x_i)$ in the interest rate market with a fixed length of time x_i from t_k to the maturity date $t_k + x_i$. We take $F(t_k, x_i)$ and $F(t_{k+1}, x_i - \Delta t)$, respectively, as $f(0, x_i)$ and $f(\Delta t, x_i)$ for each $t_k (k = 1,\cdots, J)$.

We set the change of forward rate with maturity $t_k + x_i$ from t_k to t_{k+1} by

$$\Delta F_i(t_k) = F(t_{k+1}, x_i - \Delta t) - F(t_k, x_i).$$

A sample covariance matrix is denoted by V, where

$$V_{ij} = Cov(F(t_k + \Delta t, x_i - \Delta t) - F(t_k, x_i), F(t_k + \Delta t, x_j - \Delta t) \\ - F(t_k, x_j)) / \Delta t$$

for i, j. We assume that V has rank $d \le n$. By the usual arguments of principal component analysis, the covariance matrix is decomposed as

$$V_{ij} = \sum_{l=1}^{d} e_i^l (\rho_l)^2 e_j^l,$$

where ρ_l^2 and $e^l = (e_1^l, \cdots, e_n^l)$ are the l th eigenvalue and the l th eigenvector (or principal component), respectively. We may assume that all eigenvectors are chosen such that $e_1^l > 0$ and $\rho_l > 0$. It holds that $\sum_{i=1}^{n} e_i^l e_i^h = \delta_{lh}, 1 \le l, h, \le d,$ where δ_{lh} denotes the Kronecker delta.

We assume that $\sigma(t,T)$ satisfies the relation $\sigma^l(0, x_i) = \rho_l e_i^l$. To simplify the notation, we denote $\sigma(0, x_i)$ and $\upsilon(0, x_i)$ as σ_{0i} and υ_{0i}, respectively, for $i = 1,\cdots, n$. Denoting the l th component of σ_{0i} as σ_{0i}^l, it holds that $\sigma_{0i} = (\sigma_{0i}^1, \cdots, \sigma_{0i}^d)^T$. We call $\sigma_0^l = (\sigma_{01}^l, \cdots, \sigma_{01}^l)$ the *l th volatility* for all l. Next, β_l and R_l are defined as principal component scores, by

$$\beta_l = \sum_{i=1}^{n} \sigma_{0i} \upsilon_{0i} e_i^l \tag{2.6}$$

$$R_l = \sum_{i=1}^{n} E^H \left[\frac{\Delta F_i}{\Delta t} \right] e_i^l, \tag{2.7}$$

for $l = 1,\cdots, d$, where $E^H[\cdot]$ denotes the sample mean over $k = 1,\cdots, J$.

We interpret the financial meaning of R_l according to two definitions, as follows. The *rolled trend* of the forward rate is defined as

$$E^H[\Delta F_i(t_k) / \Delta t]$$

for each $i = 1,\cdots, n$. The *observable trend* of the forward rate is defined by

$$E^H[\{F(t_{k+1}, x_i) - F(t_k, x_i)\} / \Delta t]$$

for each i. Both trends intuitively represent the historical change of the forward rate during the observation period.

Experimentally, the first principal component represents a parallel shift of the forward rate curve. Hence, for example, R_1 indicates the size of roll-down (or roll-up) in the whole curve of forward rates during the observation period. In this context, R_1 is called the *first rolled trend score*, and analogously R_l is called the *l th rolled trend score*. Additionally, the *l th observable trend score* O_l is defined by

$$O_l = \sum_{i=1}^{n} E^H \left[\frac{F(t_{k+1}, x_i) - F(t_k, x_i)}{\Delta t} \right] e_i^l.$$

Assuming that the market price of risk $\varphi(t)$ is constant during the observation period, the maximum likelihood estimation[1] of the l th market price of risk is derived as

$$\varphi_l = \frac{R_l + \beta_l}{\rho_l}, \quad l = 1,\cdots, d. \tag{2.8}$$

When the volatility is low, the term β_l takes a small value because of (2.2) and (2.6). Furthermore, ρ_l always takes a positive value. From these, φ_l is roughly determined by the l th rolled trend score. For example, when the forward rate curve rolls down, the first rolled trend score typically takes a negative value, that is, the first market price of risk is roughly negative.

Experimentally, the observable trend takes values similar to those of the rolled trend. Because of this, the value of the market price of risk is roughly explained by the observable trend. In other words, the market price

[1]The form (2.8) was originally hinted at in Yasuoka (2015a), which did not refer to the maximum likelihood estimation. It was later shown in Yasuoka (2015b), Chapter 6 that this form is the maximum likelihood estimate.

of risk reflects the average changes in the historical forward rate curve, rather than the level of the forward rate.

2.3. The Hull–White Model and Market Price of Risk

Let $\sigma > 0$ and κ be constants. With these, the short rate process in the Hull–White model is represented as

$$dr(t) = \kappa\left\{\theta(t) - r(t) + \frac{\sigma}{\kappa}\varphi_t\right\}dt + \sigma dW_t, \qquad (2.9)$$

where $\theta(t)$ is a positive process, and W_t is the one-dimensional Brownian motion under $\mathbf{P}$. Usually κ is called a mean reversion rate, that is, the rate at which $r(t)$ reverts to $\theta(t)$ is characterized by the speed κ.

Next, we define a one-dimensional volatility $\sigma(t,T)$ of $f(t,T)$ in (2.1) such that

$$\sigma(t,T) = \sigma\exp\{-\kappa(T-t)\}. \qquad (2.10)$$

From (2.1), the forward rate process $f(t,T)$ is described in the HJM framework by

$$f(t,T) = f(0,T) + \int_0^t \frac{\sigma^2}{\kappa} e^{-\kappa(T-s)}[1 - e^{-\kappa(T-s)}]ds$$

$$+\sigma\int_0^t e^{-\kappa(T-s)}\varphi_s ds + \sigma\int_0^t e^{-\kappa(T-s)}dW_s.$$

It is known that the short rate process $r(t) = f(t,t)$ satisfies (2.9); in other words, $f(t,T)$ represents a forward rate process implied from (2.9). In this context, the Hull–White model can be regarded as a special case of the HJM model. By using this forward rate process, we can apply the estimation (2.8) in the Hull–White model.

Under the same observation as in the previous section, we have already obtained the first eigenvalue $(\rho)^2$ and the first principal component $(e_1,\cdots,e_n)^T$. Naturally, we can estimate the market price of risk according to the calculation of (2.8). More specifically, applying the volatility structure (2.10) for (2.6), we can directly calculate the market price of risk. For details see Yasuoka (2015b), Chapter 8.

For clarity, we denote ρ_1 as ρ in the following and do the same with other variables. In order to approximate the first volatility in the form (2.10), consider the least-squares problem given by

$$\sum_{i=1}^n \{\rho e_i - \sigma\exp(-\kappa x_i)\}^2.$$

We determine the two parameters σ and κ by finding the solution that minimizes the above objective function and satisfies a norm-invariant condition, so that

$$(\rho)^2 = \sigma^2\sum_{i=1}^n \exp(-2\kappa x_i). \qquad (2.11)$$

Next, we define an n-dimensional vector $(\tilde{e}_1,\cdots,\tilde{e}_n)^T$ as

$$\tilde{e}_i = \frac{\sigma}{\rho}\exp(-\kappa x_i)\ ;\ i = 1,\cdots,n.$$

From (2.11), we see that $\sum_{i=1}^n (\tilde{e}_i)^2 = 1$. Regarding $(\tilde{e}_1,\cdots,\tilde{e}_n)^T$ as the first principal component, we set

$$\sigma_{0i} = \rho\tilde{e}_i\ ;\ i = 1,\cdots,n.$$

Hence, $(\sigma_{01},\cdots,\sigma_{0n})^T$ can be regarded as the first volatility component. From this, we explicitly specify the constant β of (2.6) by letting

$$\beta = -\sum_{i=1}^n \frac{\sigma^2}{\kappa}\exp(-\kappa x_i)\{1 - \exp(-\kappa x_i)\}\tilde{e}_i.$$

According to (2.7), the first rolled trend score R is given by

$$R = \sum_{i=1}^n E^H\left[\frac{\Delta F_i}{\Delta t}\right]\tilde{e}_i.$$

Substituting these into (2.8), we have the market price of risk as

$$\varphi = (R+\beta)/\rho. \qquad (2.12)$$

This simple form allows us to calculate the market prices of risk for a great many cases of historical periods.

2.4. Positive Slope Model

Some of the literature implies a negative market price of risk for long observation periods. See, for example, Cheridito *et al*. (2007), De Jong (2000), or Duffee (2002). With this, Yasuoka (2015b), Chapter 7 introduces a positive slope model to explain the negative price, which also gives useful information for empirical analysis of the market price of risk.

We assume that the forward rate is observed to have the form

$$F(t,x) = F(t,0) + ax \tag{2.13}$$

for some positive constant a. Note that this model is not assumed to be an Ito process, since this model represents a sample dataset. For a fixed time interval $\delta > 0$, we let $x_i = \delta i, i = 1, \cdots, n$. Then, it holds that

$$F(t_k, x_i) = F(t_k, 0) + ax_i \tag{2.14}$$

for all $k = 1, \cdots, J+1$ and $i = 1, \cdots, n$. Naturally, the volatility is observed as flat in x, and so we consider a one-dimensional model here. We denote the volatility by σ for some observation period, writing σ for σ^1, and do likewise with other variables for the sake of clarity. With respect to the value of the market price of risk and its sensitivity to the parameters a and σ, we have the following proposition (see the Appendix for a proof).

Proposition 2.1 *In the positive slope model, the first market price of risk is given by*

$$\varphi = \frac{O/\sqrt{n} - a}{\sigma} - \frac{1}{2}\sigma\delta(n-1), \tag{2.15}$$

where O denotes the first observable trend score. The partial derivatives of φ in a and σ are respectively given by

$$\frac{\partial \varphi}{\partial a} = -\frac{1}{\sigma}, \tag{2.16}$$

$$\frac{\partial \varphi}{\partial \sigma} = \frac{a - O/\sqrt{n}}{\sigma^2} - \frac{(n-1)\delta}{2}. \tag{2.17}$$

Since $\sigma > 0$, (2.16) shows that the market price of risk is negatively sensitive to changes in a. Since the parameter a reflects the steepness of the forward rate curve, (2.16) roughly means that steeper forward rate curves imply more strongly negative values of the market price of risk. More practically, we may rephrase this, saying that the steeper the yield curve is, the more negative the market price of risk is. The steep yield curve corresponds to a large spread of yield curve; that is, it indicates large risk premiums in the bond market. In this context, (2.16) represents a traditional interpretation between the market price of risk and bond risk premium.

The parameter σ represents the magnitude of the volatility. Then, (2.17) roughly reflects the influence of volatility on the market price of risk. However this impact is not obvious, because the sign of the right-hand side of (2.17) changes with respect to the historical data.

3. ANALYSIS AND RESULTS

3.1. Data

We use weekly U. S. Treasury yields from 2 January 1970 to 1 January 2016. Setting $\delta = 0.5$ (year) and $x_i = \delta i$ for $i = 1, 2, \cdots, 20$ $(n = 20)$, the 6-month forward rate is obtained for this period. Figure **1** shows a historical chart of the implied forward rates. Generally, the forward rates rise from the 1970s to the 1980s, and then fall from the 1980s to the present.

Figure **2** shows an average with standard deviation of the forward rates for the whole period. From this, the forward rate curve has a weakly positive slope in the short-term forward rates, and is almost flat in the long

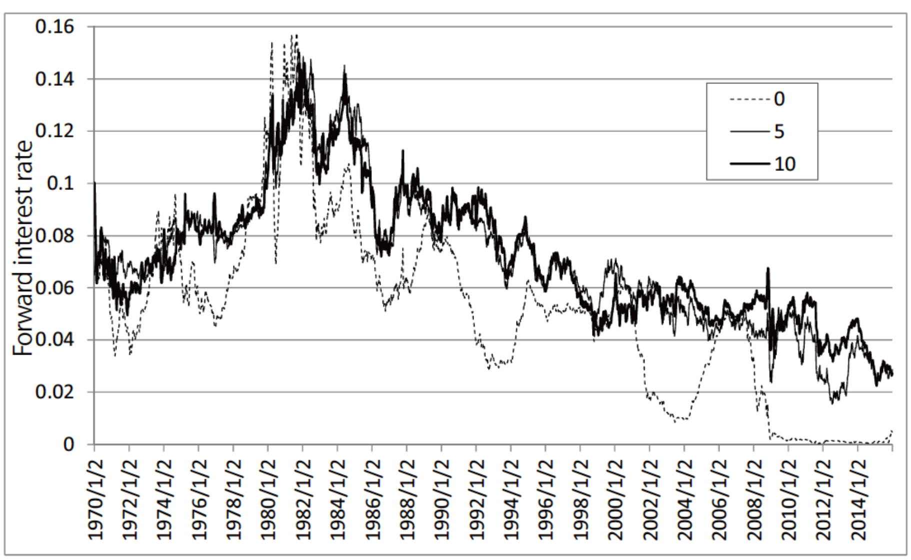

Figure 1: Implied forward rates in the U.S. Treasury market. The labels 0, 5, and 10 in indicate the forward rate over the intervals [0, 0.5], [5, 5.5] and [10, 10.5] years. Yield data were retrieved from the Board of Governors of the Federal Reserve System (2016). The forward rates were calculated by the author.

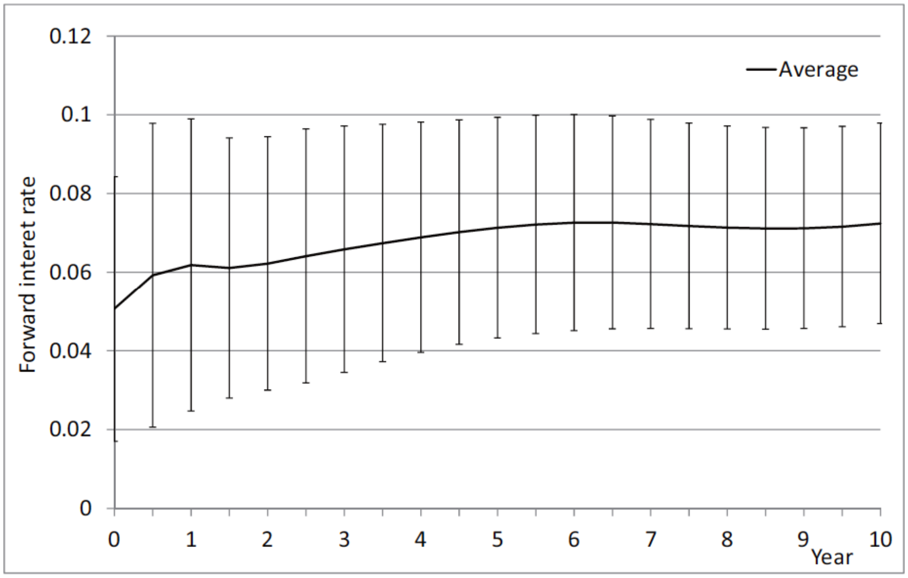

Figure 2: Average with standard deviation of the implied forward rates. Data are the same as in Figure **1**.

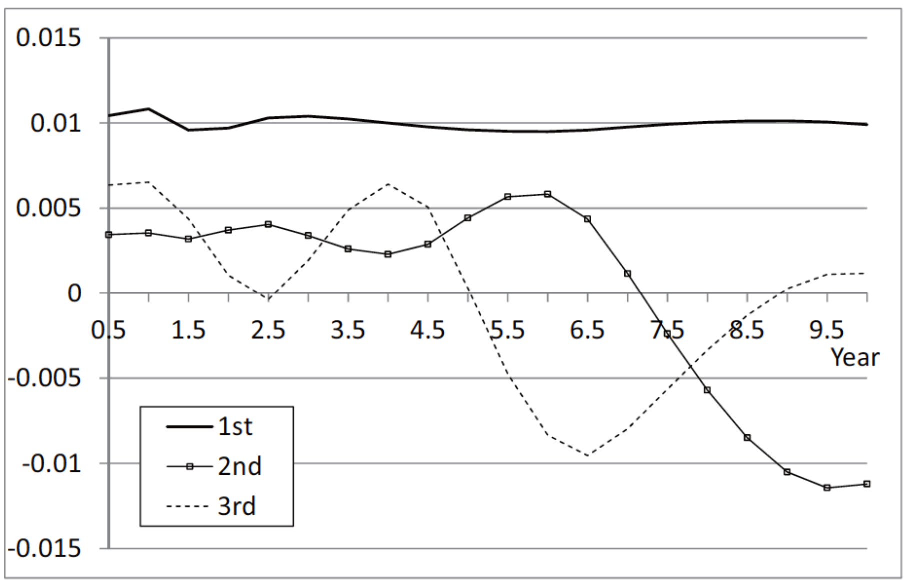

Figure 3: The first, second and third volatilities for the whole sample period (1970–2016). The third accumulated contribution rate is 84.9%. Data are the same as in Figure **1**.

term. Figure **3** exhibits the first, second, and third volatilities for the whole period. The first volatility shows a parallel change in the forward rate curve; the second volatility corresponds to the steepness change. These observations are similar to that in Litterman and Scheinkman (1991), which examines the yield changes but not the change in forward rate. Although the third volatility in forward rates is supposed to represents curvature change, like that in Litterman and Scheinkman (1991), this example shows a more complicated change. There seem to be two reasons for this difference. The first reason is that we observe the forward rate curve, which forms and changes in more complex ways than the yield curve does. The second reason is that the implied forward rate curve is not sufficiently smooth in the early period because the implied forward rate curve was not precisely observed in those early years. Regardless, this discrepancy is not a severe problem because the term structure of forward rates is studied here as a single-factor model. Specifically, our analysis works with only the first volatility.

For convenience, we regard the 6-month forward rate as the instantaneous forward rate in this paper. We calculate the market price of risk changing the length of observation period to be 1, 3, and 5 years, which are referred to as Cases A, B, and C, respectively. For example, all periods are subdivided annually, taking from the first Friday in each year to the

Table 1: Specification of observation period. The start day of each period is the first Friday of the year. For each case, the period lengths (in weeks) are set such that periods are distinct.

		Case A	Case B	Case C
Period length	**Years**	**1**	**3**	**5**
	Weeks	**52**	**156**	**261**
Beginning day	1st period	1/2/1970	1/2/1970	1/2/1970
	2nd period	1/1/1971	1/5/1973	1/3/1975
	3rd period	1/7/1972	1/2/1976	1/4/1980
	⋮	⋮	⋮	⋮
	Last period	1/2/2015	1/6/2012	1/1/2010
Number of periods		46	15	9

Friday after 52 weeks in Case A, and so on for other cases. The details are listed in Table **1**. The volatility parameters κ and σ and the market price of risk φ are calculated for each period according to the method in Section 2.3.

Figure **4** shows the chart of results for Case A, where the magnitude of volatility σ is inflated by a factor of one hundred for visibility. We see that the mean reversion rate is remarkably unstable in the ages before 1976. Although the mean reversion rate is implied from the term structure of the forward rate volatility, it is assumed that the forward rate was not as precisely analyzed in earlier years, as mentioned above. This might be one of the reasons for the observed instability. We see that the market price of risk takes values in the range -3.31 to 2.23, and changes uncertainly through the whole period. This looks to be consistent with the definition of the market price of risk as a stochastic process.

3.2. Correlation with the Market Price of Risk

Some research exists that represents the market price of risk in connection with economic state variables, such as the short rate. In contrast with that approach, (2.8) shows that the market price of risk is basically explained by the rolled trend, that is, by the average change of the forward rate curve, rather than by the level of interest rates. In connection with this interpretation, we examine the correlation between the market price of risk and other interest rates for Cases A, B, and C.

For the correlation analysis, we work with the volatility parameters κ and σ, the rolled trend score R, the 6-month yield and 10-year yield, and the term spread from the 6-month to 10-year yield. For each period of the three cases, we deal with the initial yield, average yield, yield change, and terminal yield, where the "average" is a historical average during each

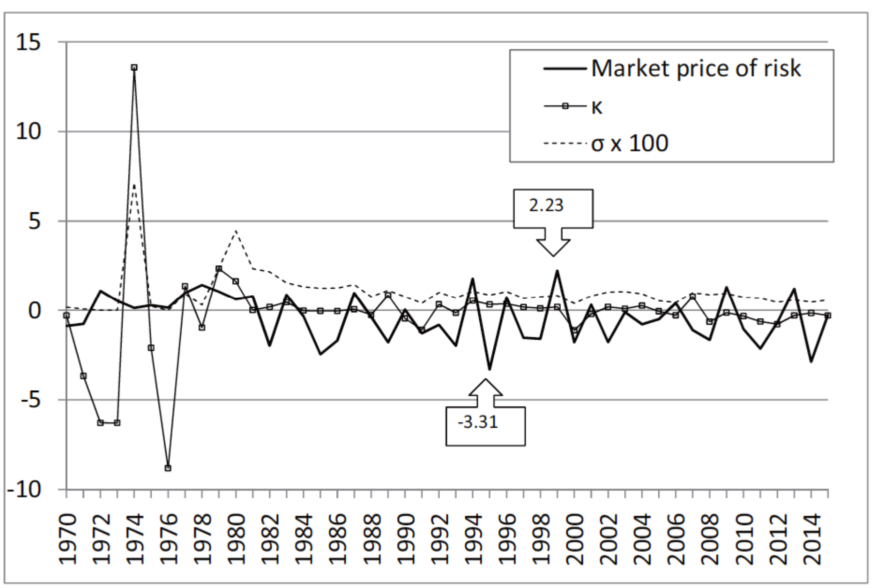

Figure 4: Chart of the market price of risk, mean reversion rate κ, and volatility size σ for Case A. Data are the same as in Figure **1**.

Table 2: Correlations between the market price of risk and interest rates. Case A1 is the first half-period of Case A, and Case A2 is the latter half of that period. Spread represents the 10-years to 6-months yield spread. The market data are the same as in Figure 1

		Case A	Case B	Case C	Case A1	Case A2
Period length (years)		**1**	**3**	**5**	**1**	**1**
Number of periods		**46**	**15**	**9**	**23**	**23**
Volatility parameters	κ	-0.019	-0.294	0.047	-0.034	0.196
		(-0.12)	(-1.11)	(-0.12)	(-0.16)	(-0.92)
	σ	0.117	0.206	0.248	0.077	0.24
		(-0.78)	(-0.76)	(-0.68)	(-0.36)	(-1.13)
Rolled trend	R	0.920***	0.933***	0.971***	0.921***	0.971***
		(-15.6)	(-9.33)	(-10.8)	(-10.8)	(-18.7)
Initial yield	6 month	0.069	0.05	0.461	-0.125	-0.08
		(-0.46)	(-0.18)	(-1.37)	(-0.58)	(-0.37)
	10 year	-0.067	-0.037	0.138	-0.360*	-0.32
		(-0.45)	(-0.13)	(-0.37)	(-1.77)	(-1.55)
	Spread	-0.327**	-0.261	-0.747**	-0.338	-0.279
		(-2.29)	(-0.97)	(-2.97)	(-1.65)	(-1.33)
Average yield	6 month	0.179	0.374	0.561*	0.096	0.024
		(-1.21)	(-1.46)	(-1.79)	(-0.44)	(-0.11)
	10 year	-0.067	0.197	0.439	-0.360*	-0.32
		(-0.45)	(-0.72)	(-1.29)	(-1.77)	(-1.55)
	Spread	-0.575***	-0.595**	-0.703**	-0.677***	-0.479**
		(-4.67)	(-2.67)	(-2.62)	(-4.21)	(-2.50)
Yield change	6 month	0.513***	0.917***	0.474	0.706***	0.388*
		(-3.96)	(-8.31)	(-1.43)	(-4.57)	(-1.93)
	10 year	0.891***	0.757***	0.421	0.905***	0.951***
		(-13.03)	(-4.18)	(-1.23)	(-9.75)	(-14.11)
	Spread	0.248*	-0.11	0.331	-0.032	0.457**
		(-1.7)	(-0.40)	(-0.93)	(-0.15)	(-2.36)
Terminal yield	6 month	0.318**	0.629**	0.670**	0.382*	0.158
		(-2.22)	(-2.92)	(-2.39)	(-1.89)	(-0.73)
	10 year	0.332**	0.509*	0.701**	0.247	0.353*
		(-2.33)	(-2.13)	(-2.6)	(-1.17)	(-1.73)
	Spread	-0.09	-0.460*	-0.28	-0.360*	0.182
		(-0.60)	(-1.87)	(-0.77)	(-1.77)	(-0.85)

*Statistically significant at a 10% significance level.
**Statistically significant at 5% significance level.
***Statistically significant at 1% significance level.

period, and the "yield change" is the difference between the terminal and initial yield for each period. Using these, we calculate the correlation coefficients between the market price of risk and these fifteen variables for Cases A, B, and C. We test the null hypothesis of no correlation between the market price of risk and these variables. Table **2** reports the correlation coefficients and the related t statistics.

3.2.1. Case A: 1-Year Observation Period

First, we examine the results of Case A. The volatility parameters κ and σ are not significantly correlated with the market price of risk. Naturally, the rolled trend score admits strongly positive correlation with the market price of risk, which is theoretically explained by (2.8).

There are no significant relations between the initial 6-month yields and the market price of risk. The initial 10-year yield shows the same result. Similarly, the average 6-month and 10-year yields show the same results. These results are explained by the observation that the market price of risk is roughly determined by the average changes in the forward rates, and not by the levels of the forward rates.

Table **2** indicates that the spreads in initial and average yields are negatively correlated with the market price of risk, at the 5% and 1% significance levels, respectively. This can be explained as "the steeper the yield curve is, the more negative the market price of risk is," as mentioned in Section 2.4. This is also explained by the traditional interpretation between the market price of risk and the risk premium. Recall that the magnitude of the volatility does not obviously affect the market price of risk, as seen in Section 2.4. This interpretation explains why the correlation with the spread is stronger than that with the volatility size σ.

For the yield change, the 6-month and 10-year yields are positively correlated with the market price of risk, at the 1% significance level. This result is natural, because the yield change corresponds to the observable trend of the forward rate, which is roughly similar to the rolled trend.

Finally, the terminal yields of the 6-month and 10-year cases are positively correlated with the market price of risk at a 10% significance level. Recall that the market price of risk is explained by the change of the forward rate curve. From this, terminal forward rates rise when the market price of risk is positive and fall when it is negative. This relation approximately holds for the terminal yields in place of the forward rates. This explains the positive correlation.

3.2.2. Cases B and C: 3- and 5-Year Observation Periods

Cases B and C give results similar to those of Case A. The differences are that we find that some variables are not significantly correlated with the market price of risk: the initial yields spread in Case B, and the 6-month and 10-year yield changes in Case C. However, the nominal direction of the correlations of these are the same as in Case A. In this context, our observation is almost stable with respect to the length of observation period.

3.2.3. Cases A1 and A2: Sensitivity to the Choice of Period

We additionally examine the sensitivity of our study with respect to the choice of period, For this, we split Case A into Cases A1 and A2. Case A1 is the first half of Case A, covering 1970–1993; Case A2 covers 1993–2016. We calculate the correlations between the same variables as was done for Case A, and the results are listed in the fourth and fifth columns of Table **2**. Comparing Case A with Cases A1 and A2, we see that the rolled trend, the spread in average yield, and the 6-month and 10-year yield changes are significantly correlated with the market price of risk at the 10% significance level or better. Although the initial yield spreads in Cases A1 and A2 are not significantly correlated with the market price of risk, the directions of the correlations are the same as in Case A. Similar results are observed for the 6-month and 10-year terminal yields. Consequently, our correlation analysis is basically insensitive to the choice of period.

4. CONCLUSION

By estimating the market price of risk, we empirically examine the relation between market price of risk and yields in the U.S. Treasury market. Specifically, we examine the correlations while changing the length of observation period.

Our results are summarized as follows. These features are insensitive to changes in length of observation period and to the choice of period.

1. The volatility parameters κ and σ have no correlation with the market price of risk.
2. The rolled trend R is strongly positively correlated with the market price of risk.
3. The initial yields and average yields are not correlated with the market price of risk.
4. The yield change and terminal yield are significantly positively correlated with the market price of risk.
5. The term spreads in initial and average yields are significantly negatively correlated with the market price of risk.

These results are theoretically explained within the mathematical model used to estimate the market price of risk.

And the above relations serve as a useful reference for risk management as well as for study of financial policy. In particular, the last result is consistent with the traditional interpretation between the market price of risk and the risk premium. Naturally, it is expected that similar findings will be observed in other major currency interest-rate markets.

APPENDIX

Proof of Proposition 2.1

Recall that the volatility is flat in x and is denoted by σ. It follows that $e_i = 1/\sqrt{n}$ and $\sigma_i = \sigma = \rho_1/\sqrt{n}$ for all i. We have from (2.2) that $\upsilon_i = \delta\sigma i$, and from (2.6) that

$$\beta = -\frac{1}{2}\delta\,\sigma^2\sqrt{n}(n-1). \tag{A.1}$$

Since

$$E^H\left[\frac{\Delta F_i(t_k)}{\Delta t}\right] = E^H\left[\frac{F(t_{k+1}, x_i) - F(t_k, x_i)}{\Delta t}\right] - a, \tag{A.2}$$

the first rolled trend score is represented by

$$R = O - a\sqrt{n}.$$

Substituting (A.1) and (A.2) into (2.8), we obtain the first market price of risk as

$$\varphi = \frac{O - a\sqrt{n} + \beta}{\rho}$$

$$= \frac{O/\sqrt{n} - a}{\sigma} - \frac{1}{2}\delta\sigma(n-1).$$

Thus, we have (2.15). From this, it follows immediately that

$$\frac{\partial\varphi}{\partial a} = -\frac{1}{\sigma}, \quad \frac{\partial\varphi}{\partial\sigma} = \frac{a - O/\sqrt{n}}{\sigma^2} - \frac{(n-1)\delta}{2}.$$

This completes the proof.

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